Statistics of extreme dynamic behaviour of marine structures

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The estimation of the probability of failure of the mooring of a marine structure needs two steps. First, a probabilistic step, which corresponds to the calculation of the distributions of the maximum forces applied to the mooring lines during a given sea-state (short-term). A second step, which uses the statistics of the sea-state conditions (climatology) to estimate the probability of failure of the line, considering all the situations that the structure will encountered during its service life (long-term). To be feasible, this statistical long-term step needs a probabilistic short term calculations, not too much costly in computing time, but of course with a sufficient accuracy in all the sea-state situations.
The modelling of behaviour of the marine structures is taken more and more complex (dynamic, nonlinear), and so the calculation of the distributions of the maxima during a given sea-state needs adapted methodologies. The roll motion and the low frequency movement of a floating marine structure will be given as examples.
Comparisons between different methods and Gaussian hypotheses will be commented.

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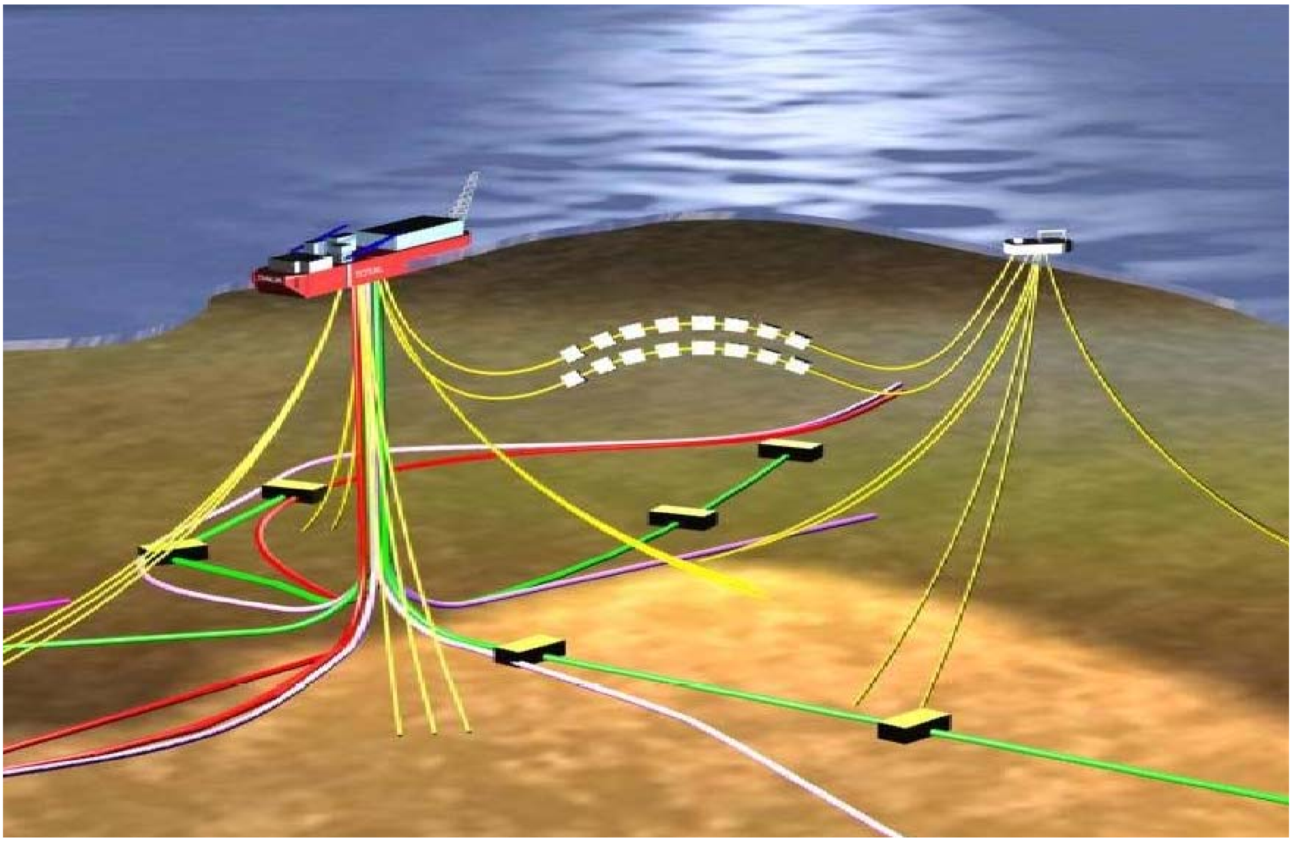
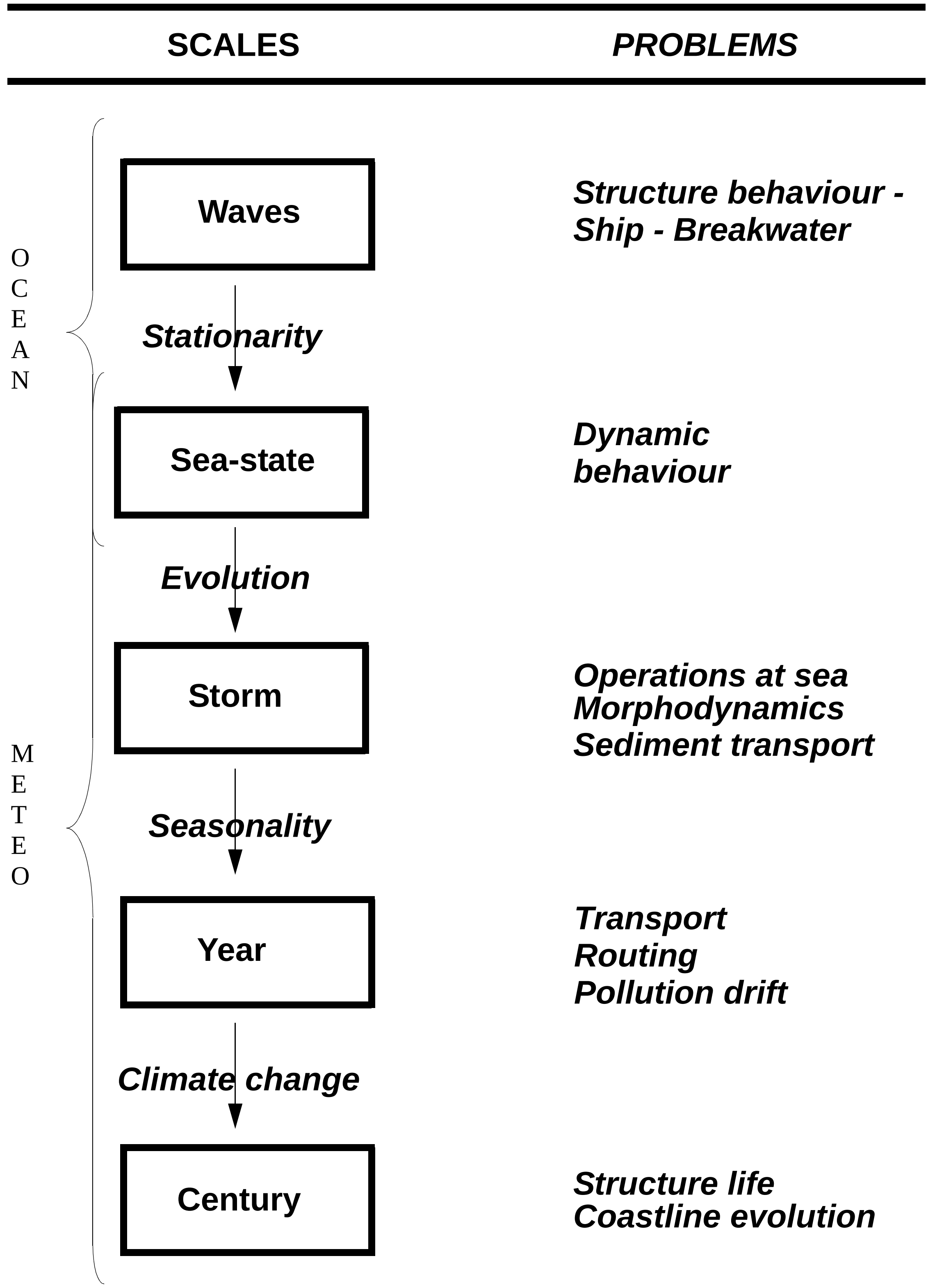


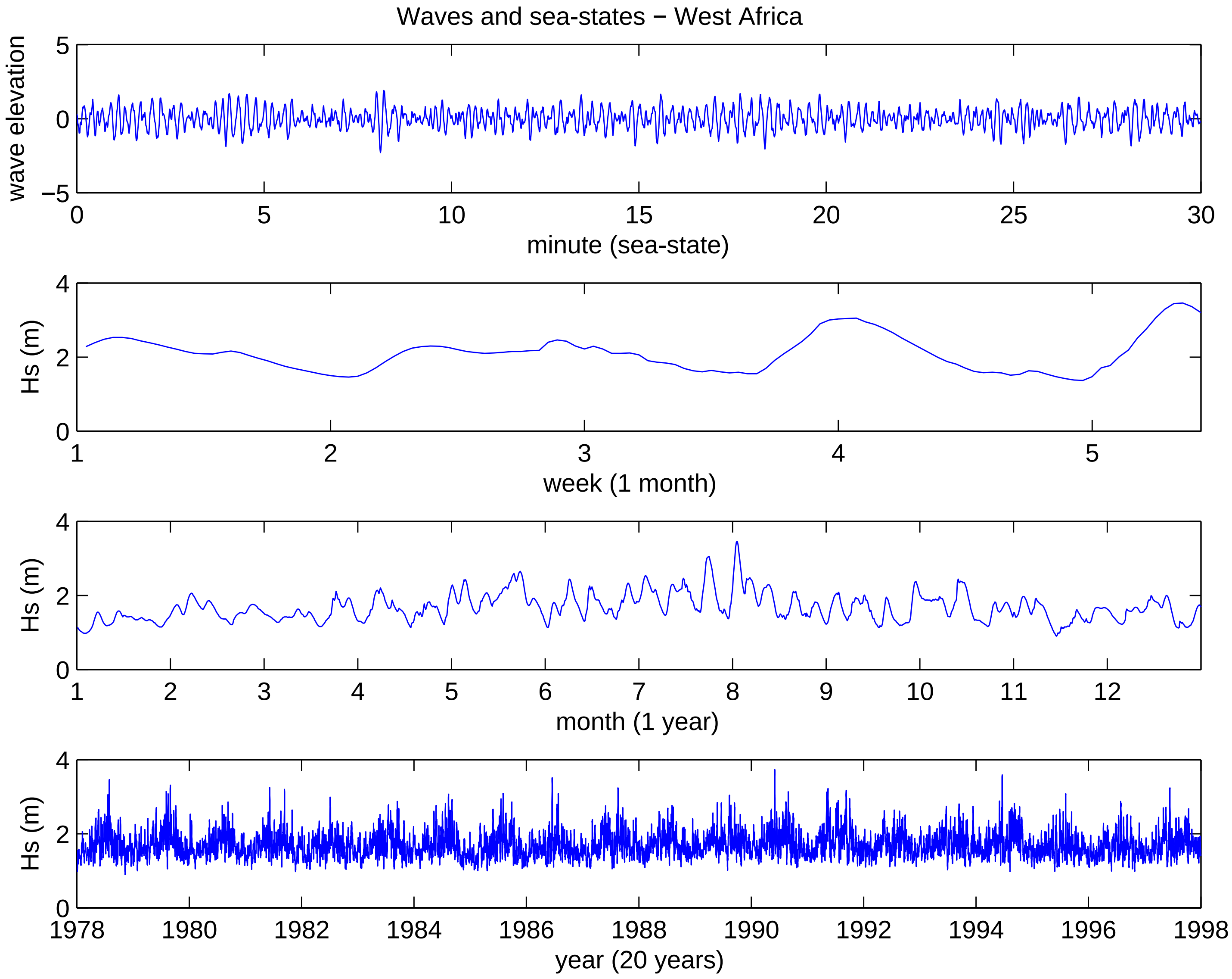
- **Combined effects of wind, waves, current**
- **Complex nonlinear dynamic behavior**
- **Design for extreme conditions**

Introduction of statistics and probabilities

- Random phenomena: wind, waves, current -> statistics/climatologies
- Time evolution (short/medium/long term-> process, dependence
- Simultaneity -> joint extremes
- Design -> probability of failure $P(X_{max}^{life} > x_r)$

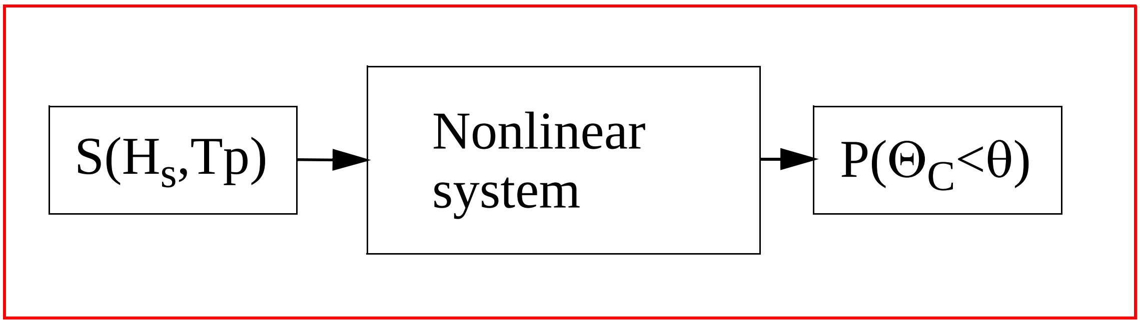
FIGURE 1. Scales for metocean conditions




Waves and sea-states − West Africa
wave elevation
5
0
−5
0
5
10
15
20
25
30
minute (sea-state)
Hs (m)
4
2
0
1
2
3
4
5
week (1 month)
Hs (m)
4
2
0
1
2
3
4
5
6
7
8
9
10
11
12
month (1 year)
Hs (m)
4
2
0
1978
1980
1982
1984
1986
1988
1990
1992
1994
1996
1998
year (20 years)

Probability of failure $P(X^{life}_{max} > x_r) < 10^{-?}$

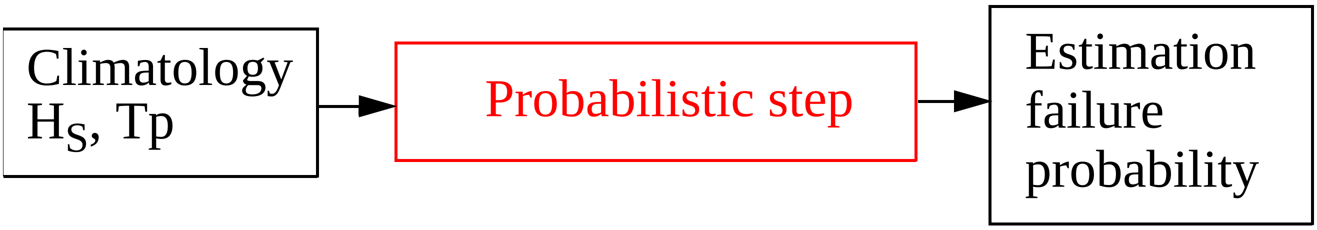
Probabilistic step (short term):



Probabilistic step

Decomposition in stationary state (sea-states ~3hours)

Statistical step (long term):



$$P(S \leq s) = \int P(S \leq s | \Sigma = \sigma) p_{\Sigma}(\sigma) d\sigma \qquad (1)$$

$$P(X^{\text{sea-state}}_{\max} \leq x) = \int P(X^{\text{sea-state}}_{\max} \leq x | H_S) p_{H_S}(h) dh \qquad (2)$$

$$P(X^{\text{life}}_{\max} \leq x) = P(X^{\text{sea-state}}_{\max} \leq x)^{\#\text{sea-state/life}} \qquad (3)$$

- **H_s of the climatology << H_s of failure -> extrapolation**
- **H_s dependence -> extremal index**

Probabilistic step (short term):

- Rice series (factorial moments)

$$P(X_{\max}^{\text{sea-state}} > x) = P(X_{t_0} > x) + P(N \geq 1 | X_{t_0} \leq x)$$

$$= P(X_{t_0} > x) + P(N = 1) + P(N = 2) + P(N = 3) + \ldots$$

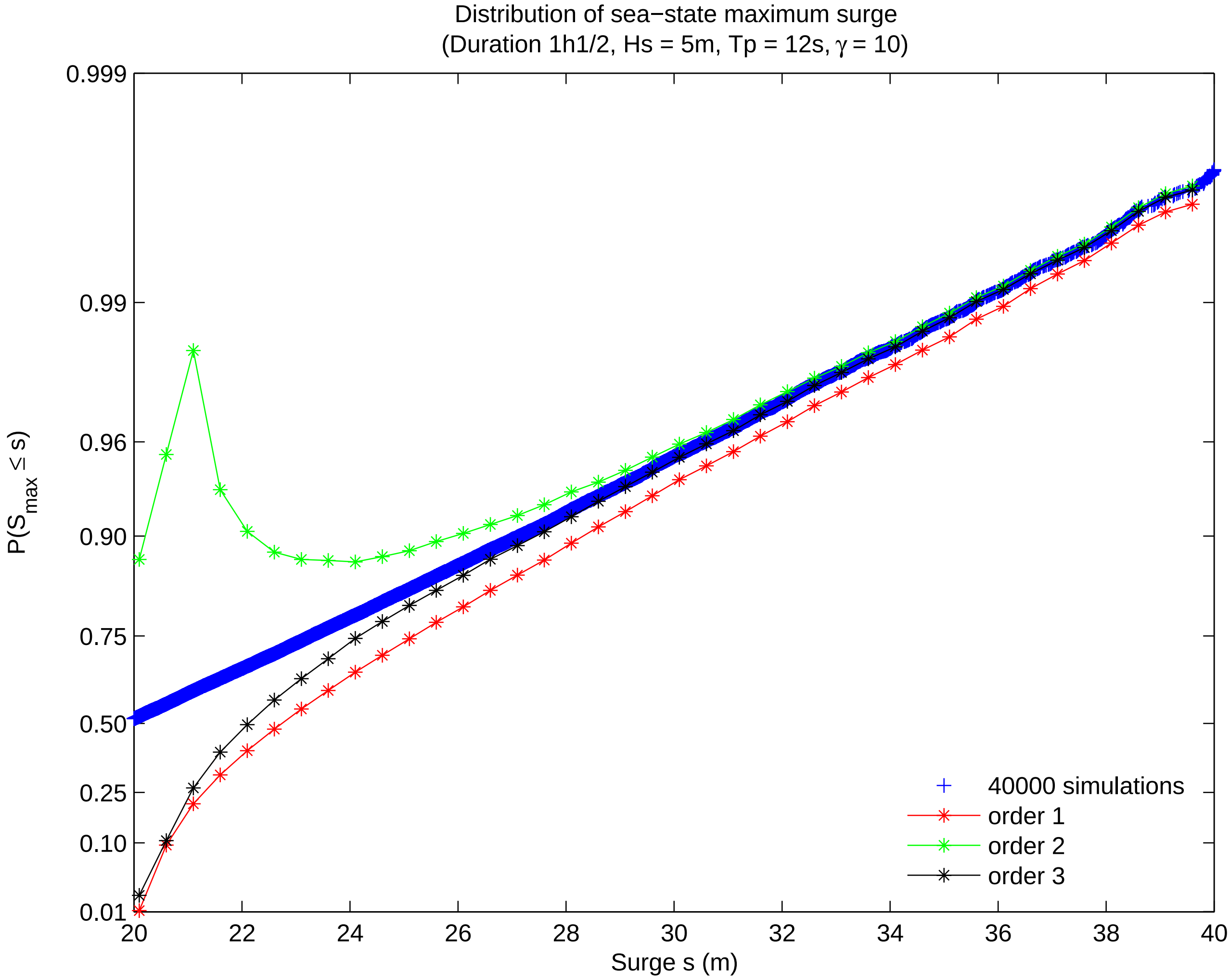
$$= P(X_{t_0} > x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \mathbb{E}(N(N-1)\ldots(N-n+1))$$

Rem. Azaïs, Wschebor -> Results for X Gaussian process

- Davies bound (x high, sea-state of short duration)

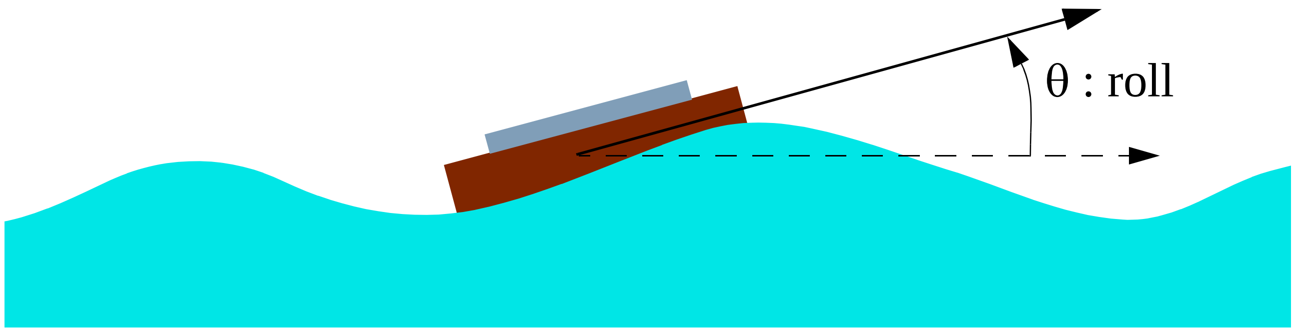
$$P(X_{\max}^{\text{sea-states}} > x) = \mathbb{E}(N) \qquad (5)$$

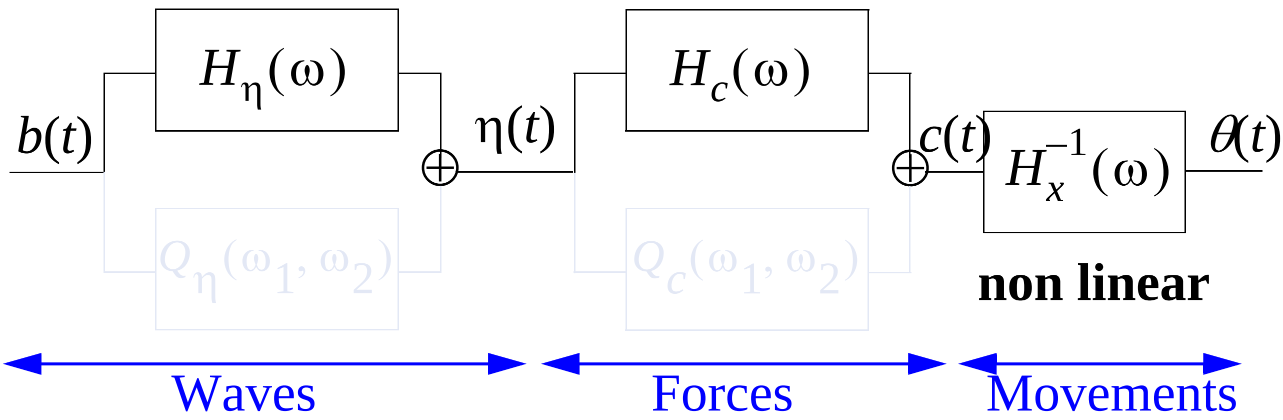
to be verified for all sea-states participating in the long term step



Gumbel plot, 30m = 4 * standard-deviation

Extreme roll





Model:

$$I\ddot{\theta}(t) + T\dot{\theta}(t)\left|\dot{\theta}(t)\right| + K\theta(t) = C(t) \qquad (1)$$

Excitation of roll, moment:

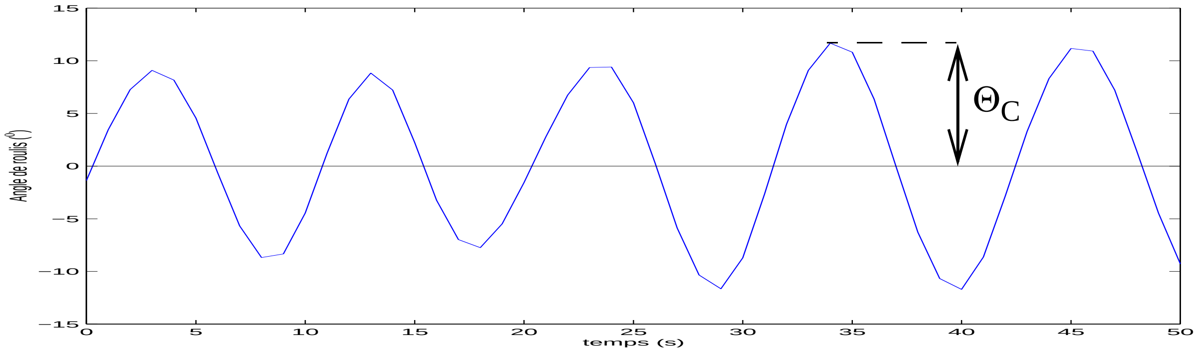
$$C(t) = h_C(t) \otimes \eta(t) \qquad (2)$$

Wave elevation:

$$\eta(t) = h_\eta(t) \otimes b(t) \quad \text{with } b(t) \text{ Gaussian white noise and } |H(\omega)|^2 = S_\eta(\omega) \qquad (3)$$

Distribution of maxima :

$$P(\Theta_C < \theta) \qquad (4)$$



Narrow band process:

$$P(\Theta_C > \theta) = \int_\theta^\infty f_{\Theta_C}(\theta)d\theta = \frac{\mathbb{E}(N_\theta)}{\mathbb{E}(N_0)} = \frac{\mu^+(\theta)T}{\mu^+(0)T} \qquad (5)$$

Rice, stationary case:

$$\mu^+(\theta) = \int_0^\infty \dot{\theta} f_{\Theta\dot{\Theta}}(\theta, \dot{\theta})d\dot{\theta} \qquad (6)$$

if joint distribution is difficult to obtain:

- **Independence hypothesis:**

$$\mu^+(\theta) = f_\Theta(\theta)\int_0^\infty \dot{\theta} f_{\dot{\Theta}}(\dot{\theta})d\dot{\theta} \qquad (7)$$

- **Projection method (perturbation of independence)**

$$f_{\Theta\dot{\Theta}}(\theta, \dot{\theta}) = f_{\Theta}(\theta) f_{\dot{\Theta}}(\dot{\theta}) s(\theta, \dot{\theta}) \tag{8}$$

- **Edgeworth, Gram-Charlier (Gaussian perturbation)**

Marginal and joint probability laws

- Simulation (Monte Carlo)
- Equivalent Linearisation
- Volterra, Wiener kernel
- Fokker-Planck equation
  - Exact solution
  - Equivalent linear system
  - Stochastic averaging
  - Numeric (FE)
- Linearize&Match [Armand,Duthoit]

Restricting hypotheses

- **White noise input**

$$I\ddot{\theta}(t) + T\dot{\theta}(t)\left|\dot{\theta}(t)\right| + K\theta(t) = C(t) \qquad (9)$$

replaced by:

$$\begin{aligned} I\ddot{\theta}(t) + T\dot{\theta}(t)\left|\dot{\theta}(t)\right| + K\theta(t) - C(t) &= 0 \\ \ddot{C}(t) + 2\xi_\eta\omega_\eta\dot{C}(t) + \omega_\eta^2 C(t) &= B(t) \end{aligned} \qquad (10)$$

- **Damping in** $\dot{\theta}(t)\left|\dot{\theta}(t)\right|$

$$I\ddot{\theta}(t) + T\dot{\theta}(t)\left|\dot{\theta}(t)\right| + K\theta(t) = C(t) \qquad (11)$$

can be replaced by:

$$I\ddot{\theta}(t) + \alpha\dot{\theta}(t) + \Xi\dot{\theta}(t)^3 + K\theta(t) = C(t) \qquad (12)$$

- **Weak damping**

Variations of $S_\eta(\omega)$ are very small in the frequency band of the response.

Input can be considered as a white noise.

Linearize&Match

• Equivalent linear system

$$I\ddot{\theta}(t) + T\dot{\theta}(t)|\dot{\theta}(t)| + K\theta(t) = C(t) \tag{13}$$

equivalent linear system:

$$I\ddot{\theta}(t) + \alpha_L\dot{\theta}(t) + K\theta(t) = C(t) \tag{14}$$

minimize for α_L

$$\mathbb{E}(\varepsilon^2), \text{ with } \varepsilon = T\dot{\theta}(t)|\dot{\theta}(t)| - \alpha_L\dot{\theta}(t) \tag{15}$$

one uses a Gaussian closure on $\dot{\theta}$:

$$\alpha_L = T\frac{4}{\sqrt{2\pi}}\mathbb{E}(\dot{\theta}^2)^{1/2} \tag{16}$$

• Cubic equivalent system

$$I\ddot{\theta}(t) + T\dot{\theta}(t)|\dot{\theta}(t)| + K\theta(t) = C(t) \tag{17}$$

replaced by:

$$I\ddot{\theta}(t) + \alpha\dot{\theta}(t) + \Xi\dot{\theta}(t)^3 + K\theta(t) = C(t) \tag{18}$$

$$\alpha = T\sqrt{\frac{2\,\mathbb{E}(\dot{\theta}^2)}{\pi}}, \quad \Xi = T\sqrt{\frac{2}{9\pi\,\mathbb{E}(\dot{\theta}^2)}} \tag{19}$$

• Sequence of linear equivalent systems

$$I\ddot{\theta}_i(t) + (\alpha + \beta_i \Xi \, \mathbb{E}(\dot{\theta}_i^2(t)))\dot{\theta}_i(t) + K\theta_i(t) = C(t) \qquad (20)$$

who verify:

$$\mathbb{E}(\theta_i(t)^{2i}) = \mathbb{E}(\theta(t)^{2i}) \qquad (21)$$

in using a Gaussian closure.

- β_i identification

$$\theta = \theta_{(1)} + \theta_{(3)} + \theta_{(5)} + \ldots \qquad (22)$$

$$\mathbb{E}(\theta^{2n}) = \mathbb{E}(\theta_{(1)}^{2n}) + 2n\,\mathbb{E}(\theta_{(1)}^{2n-1}\theta_{(3)}) + O(\theta_{(1)}^{2(n+2)}) \qquad (23)$$

$$\beta = 3 + 2(i-1)\frac{\int h(\tau) R^3_{\theta_{(1)}\dot{\theta}_{(1)}}(\tau)d\tau}{\mathbb{E}(\dot{\theta}_{(1)}^2)\,\mathbb{E}(\theta_{(1)}^2)\int h(\tau) R_{\theta_{(1)}\dot{\theta}_{(1)}}(\tau)d\tau} \qquad (24)$$

- Calculus of $\mathbb{E}(\theta^{2i})$

The $\mathbb{E}(\dot{\theta}_i^2(t))$ are calculated with an iteratif scheme:

$$\mathbb{E}(\dot{\theta}_i^2) = \int_{-\infty}^{\infty} \omega^2 \left| \frac{1}{-I\omega^2 + i\omega(\alpha + \beta_i \Xi \, \mathbb{E}(\dot{\theta}_i^2)) + K} \right|^2 S_{CC}(\omega)d\omega$$

then

$$\mathbb{E}(\theta_i^{\,2}) = \int_{-\infty}^{\infty} \left| \frac{1}{-I\omega^2 + i\omega(\alpha + \beta_i \Xi\, \mathbb{E}(\dot{\theta}_i^{\,2})) + K} \right|^2 S_{CC}(\omega) d\omega$$

which give the $\mathbb{E}(\theta_i^{\,2i})$ in using Gaussianity of the ouput of linear systems.

- **Roll maxima probability law**
  - **Marginal law**

Maximum entropy

$$max\left(-\int_{-\infty}^{\infty} f_\Theta(\theta) log(f_\Theta(\theta)) d\theta \right) \tag{27}$$

with moment constraints

$$\int_{-\infty}^{\infty} f_\Theta(\theta)\theta^{2i} d\theta = \mathbb{E}(\theta_i^{\,2i}) \tag{28}$$

- Maxima

$$P(\Theta_C > \theta) = \frac{\mu^+(\theta)}{\mu^+(0)} = \frac{\int_0^\infty \dot\theta f_{\Theta\dot\Theta}(\theta, \dot\theta) d\dot\theta}{\int_0^\infty \dot\theta f_{\Theta\dot\Theta}(0, \dot\theta) d\dot\theta} \qquad (29)$$

in using

$$f_{\Theta\dot\Theta}(\theta, \dot\theta) = f_\Theta(\theta) f_{\dot\Theta}(\dot\theta) \qquad (30)$$

or

$$f_{\Theta\dot\Theta}(\theta, \dot\theta) = f_\Theta(\theta) f_{\dot\Theta}(\dot\theta) s(\theta, \dot\theta) \qquad (31)$$

where the $s(\theta, \dot\theta)$ are estimated in introducing the joint moment $\mathbb{E}(\theta^2\dot\theta^2)$ calculated from the fourth order linear equivalent system

$$\mathbb{E}(\theta^2\dot\theta^2) = \mathbb{E}(\theta_{i=2}^2 \dot\theta_{i=2}^2) = \mathbb{E}(\theta_{i=2}^2)\, \mathbb{E}(\dot\theta_{i=2}^2) \qquad (32)$$

- **«Exact» closure**

New equivalent linear systems by a closure with $f_{\dot\Theta}(\dot\theta)$:

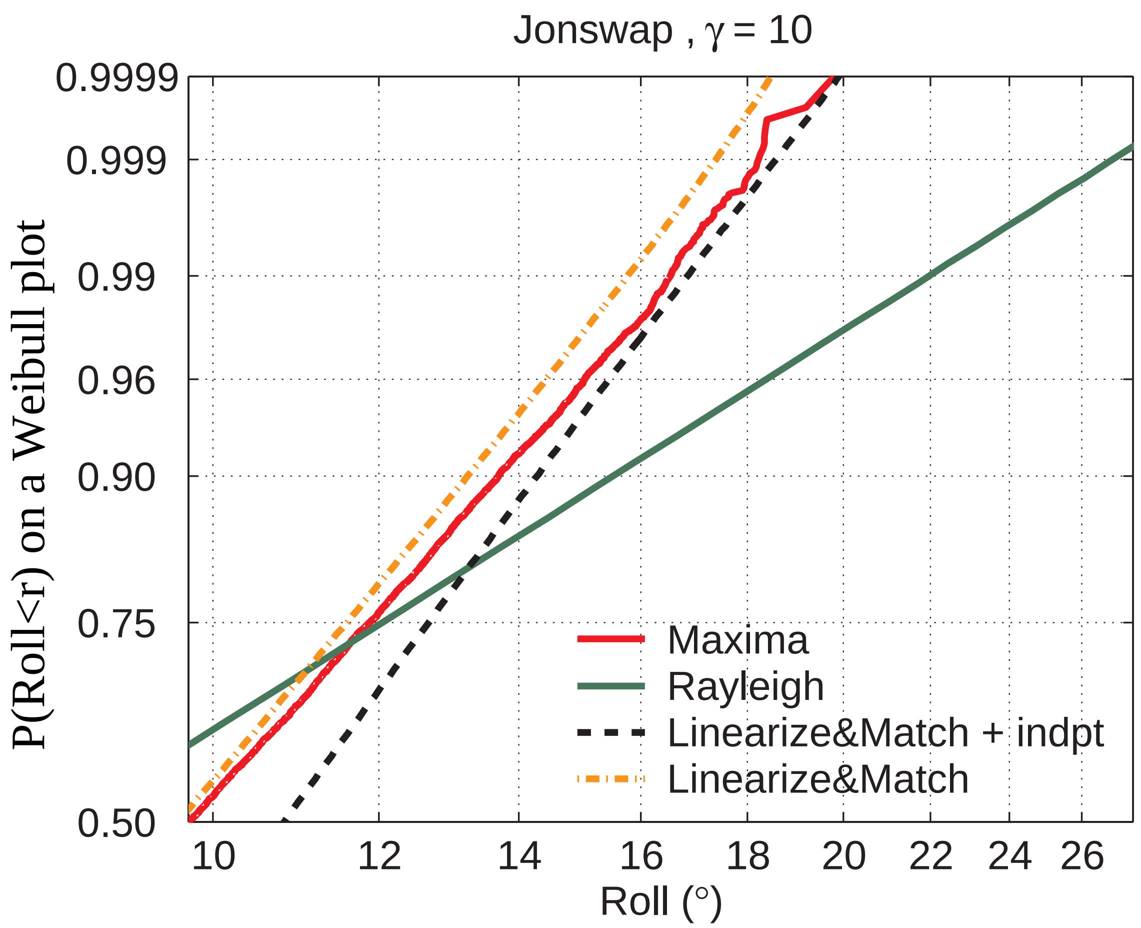
$$I\ddot\theta(t) + \alpha_L^{(2)}\dot\theta(t) + K\theta(t) = C(t) \qquad (33)$$

which give new $\mathbb{E}(\theta_i^2)$ et $\mathbb{E}(\dot\theta_i^2)$, until convergence (fast).

Moments Linearize&Match

"Exacts" moments (black numbers) $\gamma = 10,\ T_p = T_r.$
Linearize&Match (white numbers)
Linearize&Match iterated (blue numbers).

$\mathbb{E}(\theta^k\dot{\theta}^l)$ normalised	$l(\dot{\theta})$ 0		2		4		6	
$k(\theta)$ 0			16.53	14.65 16.41	2.02	2.16	5.43	6.14
2	52.37	45.96 51.80	0.69	0.72				
4	2.06	2.14						
6	5.66	6.18						



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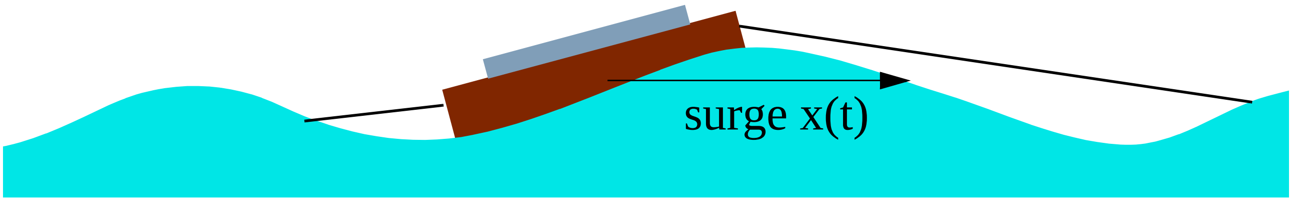
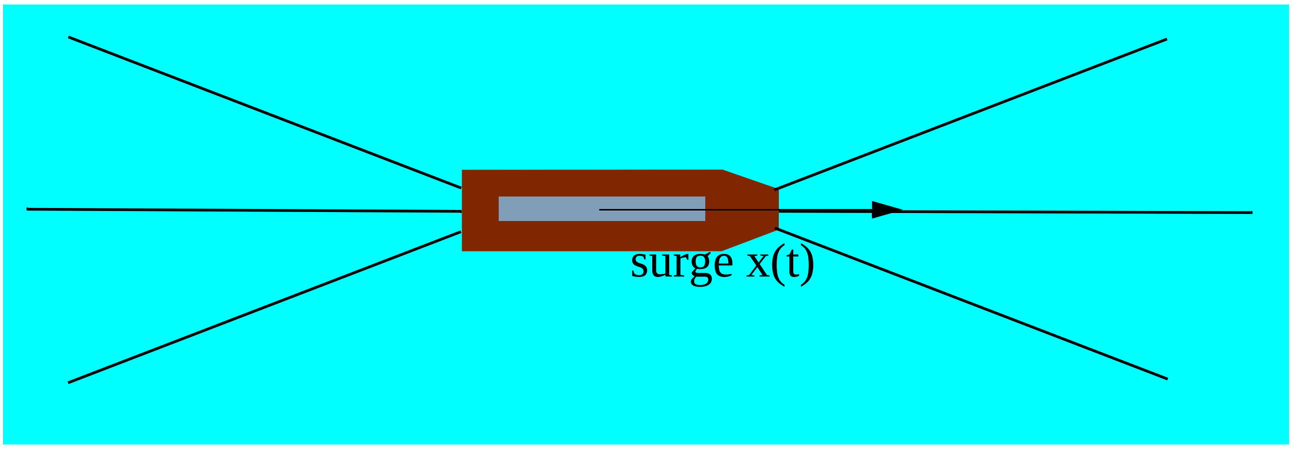
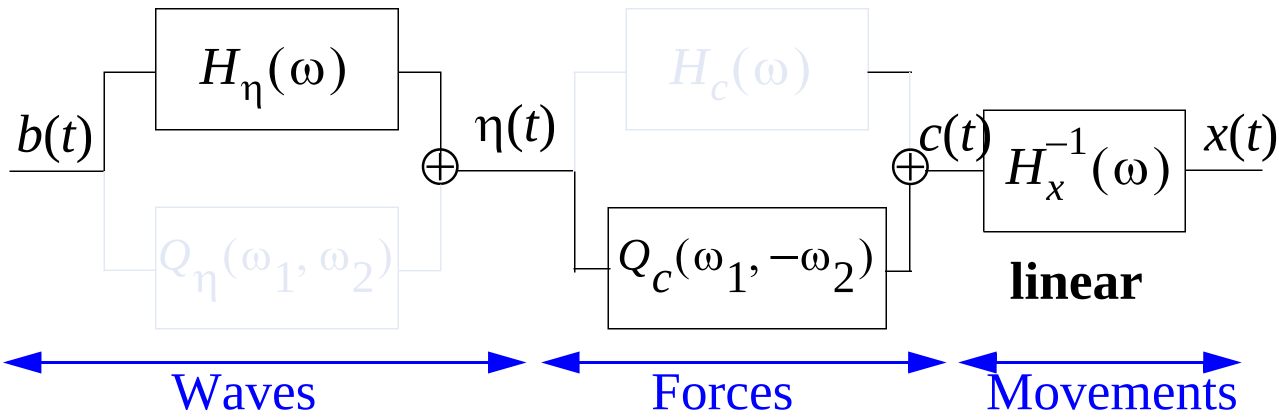
Extremes of low frequency movements







Global transfer between b(t) and x(t):

$$Q_T(\omega_1, -\omega_2) = H_\eta(\omega_1)H_\eta(-\omega_2)Q_c(\omega_1, -\omega_2)H_x^{-1}(\omega_1 - \omega_2)$$

i.e.:

$$x(t) = 2\int_0^\infty\int_0^\infty Q_T(\omega_1, -\omega_2)B(\omega_1)e^{j\omega_1 t}\overline{B(\omega_2)}e^{-j\omega_2 t}d\omega_1 d\omega_2 \quad (35)$$

Eigen-decomposition of Q_T:

$$x(t) = \frac{1}{2}\sum_{i=1,n} \lambda_i(z_i(t)^2 + \tilde{z}_i(t)^2)$$

with $z_i(t)$ and $\tilde{z}_i(t)$ standard Gaussian processes (36)

$\tilde{z}_i(t)$ Hilbert transform of $z_i(t)$

and $\mathbb{E}(z_i z_j) = \mathbb{E}(z_i \tilde{z}_i) = \mathbb{E}(\tilde{z}_i \tilde{z}_j) = \mathbb{E}(z_i \tilde{z}_j) = 0$

- **Calculus of $\mathbb{E}(N_\beta)$**
  - **Rice**

$$\mu^+(\beta) = \int_0^\infty \dot{x} f_{x\dot{x}}(\beta, \dot{x})d\dot{x} \quad (37)$$

Gaussian hypothesis, independence, projection method (with moments calculated from $x(t)$, EQ 36).

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- Lindgren (1980)

$$x(t) - \beta = \frac{1}{2}\sum_{i=1,n} \lambda_i(z_i(t)^2 + \tilde{z}_i(t)^2) - \beta = G(\beta) = 0 \quad (38)$$

$$\mu(\beta) = \frac{\beta^{n-1}}{(2\pi)^{n/2}} \int_{\partial G} \mathbb{E}(\|n(z)^T \dot{z}(t)\| \,|\, z(t) = \beta x) e^{-\frac{\beta}{2} x^T x} ds(x))$$

Breitung proposed une asymptotic approximation valid if the number of points of $G(\beta)$ at a minimal distance of 0 is finite.

- Hagberg (2004)

Particular case of quadratic forms

$$\sum_{i=1,n} \gamma_i w_i(t)^2 \quad (40)$$

and $\gamma_1 \le \ldots \le \gamma_k < \gamma_{k+1} = \ldots = \gamma_n$ (41)

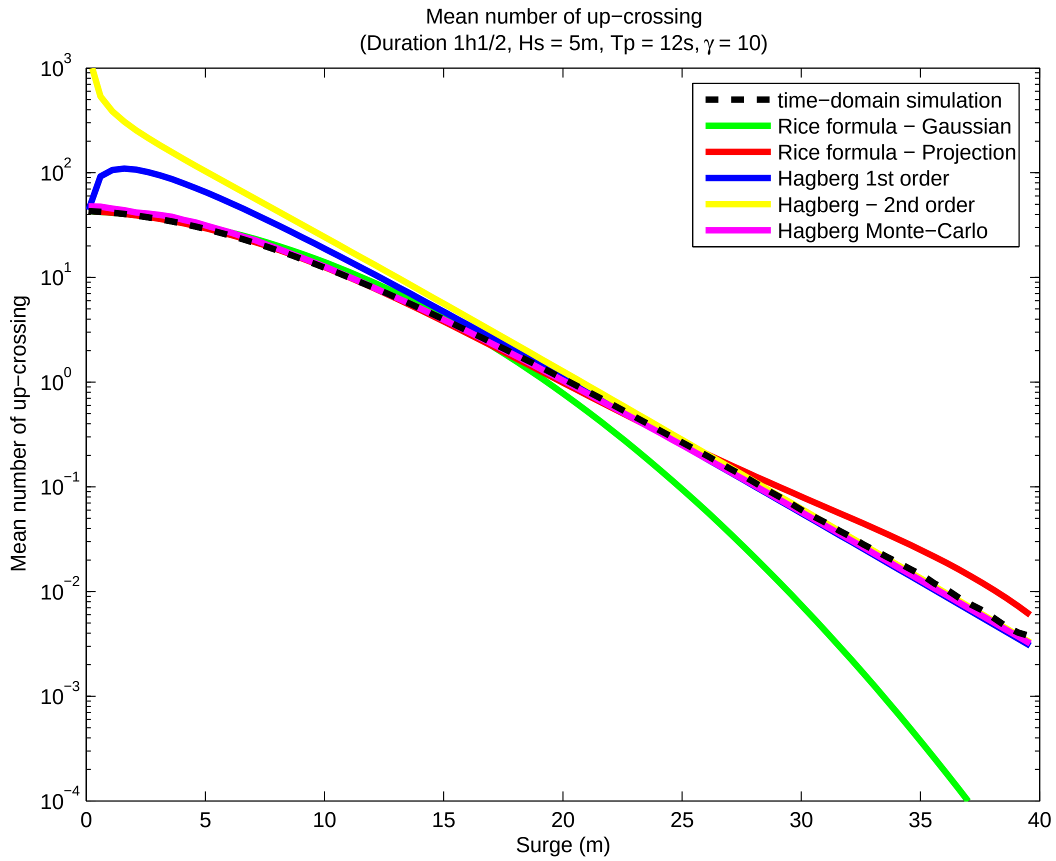
$$\mu(\beta^2) = A(\beta) \int_{r \in S_{n-k-1}} I(r, \beta) ds(r) \quad (42)$$

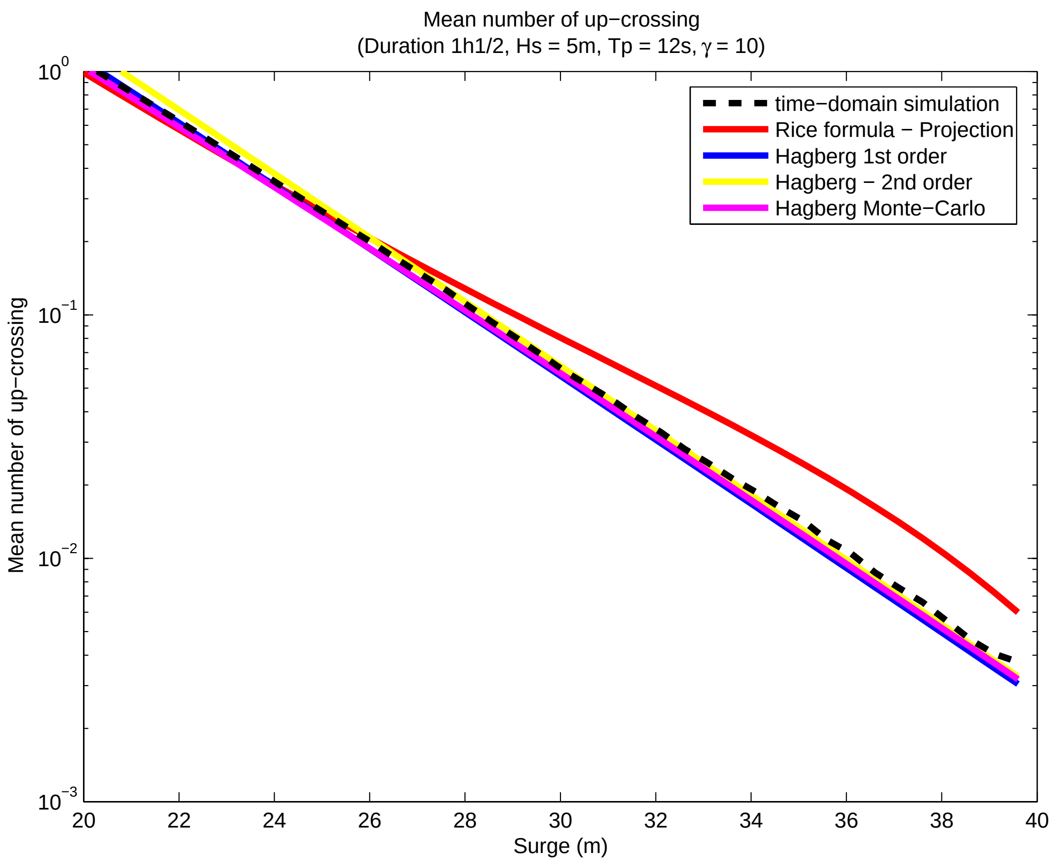
$$I(r, \beta) = \int_{s^T \Gamma_k s < \beta^2} Q(s, r, \beta) e^{-\frac{1}{2} s^T (I_k - \Gamma_k) s} ds \quad (43)$$

- asymptotic development

$$\mu(\beta^2) = A(\beta) \sum_j c_j \frac{1}{\beta^{2j}}, \; c_0, c_1 \quad (44)$$

- Monte-Carlo integration


Mean number of up−crossing
(Duration 1h1/2, Hs = 5m, Tp = 12s, γ = 10)
Mean number of up−crossing
Surge (m)
time−domain simulation
Rice formula − Gaussian
Rice formula − Projection
Hagberg 1st order
Hagberg − 2nd order
Hagberg Monte−Carlo


Mean number of up−crossing
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Mean number of up−crossing
Surge (m)
time−domain simulation
Rice formula − Projection
Hagberg 1st order
Hagberg − 2nd order
Hagberg Monte−Carlo

Improvements

- Rice series, non Gaussian case
- Comparisons with higher damping (non Gaussian vicinity)
- Addition of a linear part:

$$x(t) = \frac{1}{2}\sum_{i=1,n} \lambda_i (z_i(t)^2 + \tilde{z}_i(t)^2) + \sum_{i=1,n} \gamma_i z_i(t)$$

- $\mathbb{E}(N)$ for more complex systems