Extreme Wave Crests in Space and Time

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Different parts of this work with Prof. Rychlik and Dr. Hagberg

Sea Models

Sea surface elevation is modelled by means of

- Linear model, $W(\boldsymbol{\tau}) = W(x, y, t)$ - Gaussian random field that is homogeneous (invariant under translation but not rotation)

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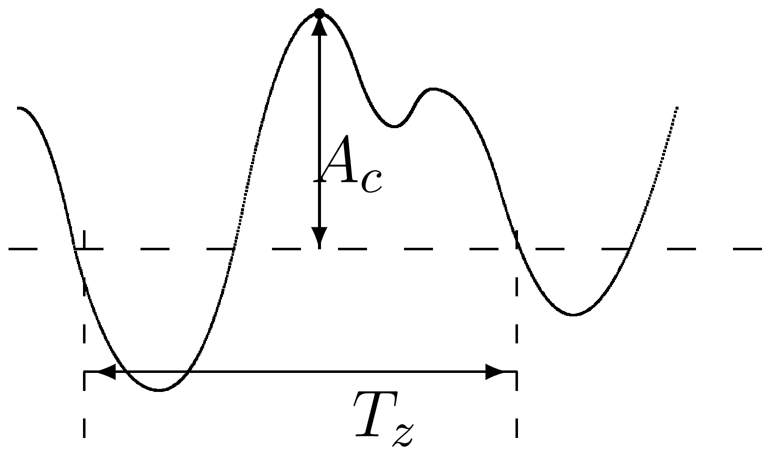
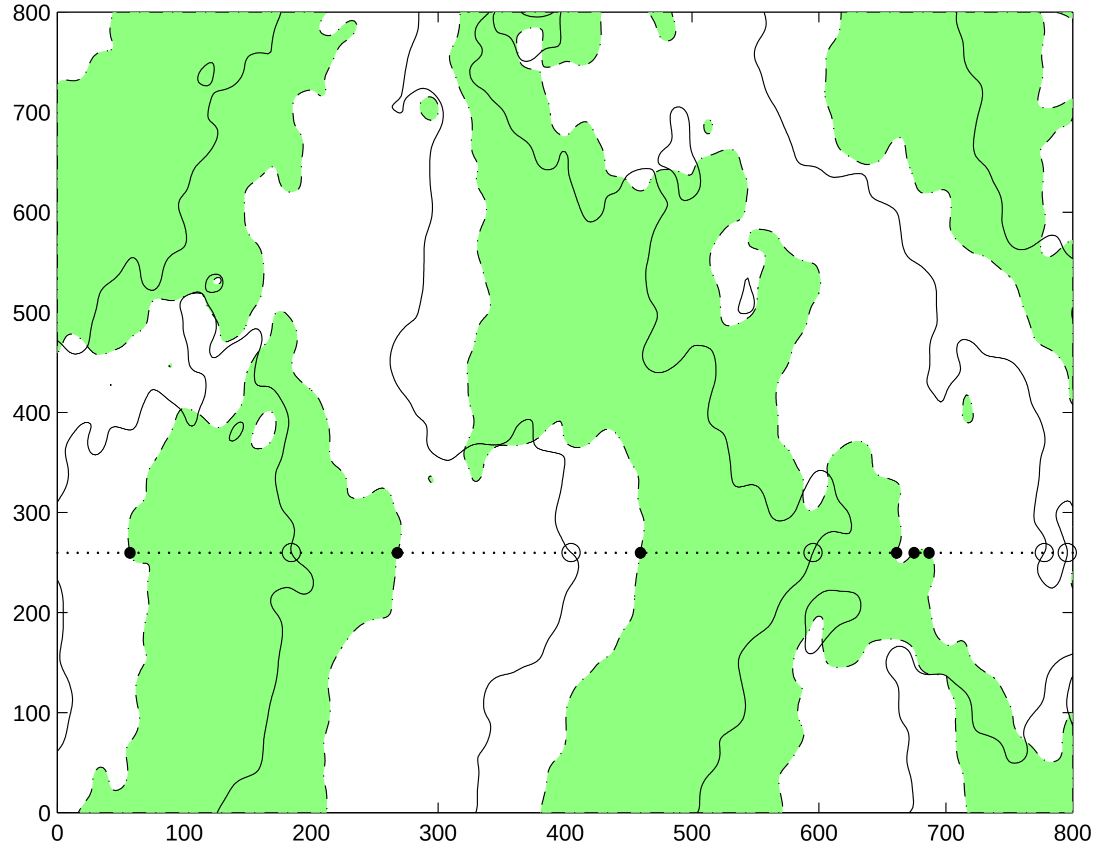
- Linear model, $W(\boldsymbol{\tau}) = W(x, y, t)$ - Gaussian random field that is homogeneous (invariant under translation but not rotation)
- Second order model, $W(\boldsymbol{\tau}) = W_l(\boldsymbol{\tau}) + W_q(\boldsymbol{\tau})$ - the linear field plus a quadratic correction term

Wave characteristics in time



Tz: wave period; Ac: crest height

Wave characteristics in space



Spatio-temporal wave characteristics

- $H_s = 4\sqrt{\lambda_{000}}$, significant wave height
- $T_z = 2\pi\sqrt{\frac{\lambda_{000}}{\lambda_{002}}}$, average zero down-crossing wave period
- $L_x = 2\pi\sqrt{\frac{\lambda_{000}}{\lambda_{200}}}$, average zero down-crossing wave length along x-axis
- $L_y = 2\pi\sqrt{\frac{\lambda_{000}}{\lambda_{020}}}$, average zero down-crossing wave length along y-axis
- $N = \frac{X}{L_X}\frac{Y}{L_Y}\frac{T}{L_T}, N_T = \frac{X}{L_X}\frac{Y}{L_Y}, N_X = \frac{Y}{L_Y}\frac{T}{L_T}, N_Y = \frac{X}{L_X}\frac{T}{L_T}$

Spatio-temporal wave characteristics

- $V_{drift} = (V_x, V_y) = (L_x/T_z, L_y/T_z) = (\sqrt{\frac{\lambda_{002}}{\lambda_{200}}}, \sqrt{\frac{\lambda_{002}}{\lambda_{020}}})$
- $\mathbf{v}_{pr} = (v_x, v_y) = \left(-\frac{\lambda_{101}}{\lambda_{200}}, -\frac{\lambda_{011}}{\lambda_{020}}\right)$
- $\alpha_{xt} = -\frac{v_x}{V_x} = \frac{\lambda_{101}}{\sqrt{\lambda_{200}\lambda_{002}}}$ and $\alpha_{yt} = -\frac{v_y}{V_y} = \frac{\lambda_{011}}{\sqrt{\lambda_{020}\lambda_{002}}}$
  - $\alpha_{xt}, \alpha_{yt} \in [-1, 1]$
  - $\alpha_{xt}^2 \approx 1$ for narrow-band sea
  - $|\alpha_{xt}| = 1$ for a sea drifting along x-axis
  - $\alpha_{xt} = 0$ and $\alpha_{yt} = 0$ for confused sea (no organised movement of waves)

Wave crest height and wave velocities

$$
\begin{aligned}
P(\max_{\boldsymbol{\tau}\in\mathbf{S}} W(\boldsymbol{\tau}) \geq u) \leq & \left(\frac{X}{L_x} + \frac{Y}{L_y} + \frac{T}{T_z}\right) e^{-\frac{8u^2}{H_s^2}} \\
+ & \sqrt{2\pi}\left(N_t + N_y\sqrt{1-\alpha_{xt}^2} + N_x\sqrt{1-\alpha_{yt}^2}\right)\frac{4u}{H_s} e^{-\frac{8u^2}{H_s^2}} \\
+ & 2\pi N\sqrt{1-\alpha_{xt}^2-\alpha_{yt}^2}\frac{16u^2}{H_s^2} e^{-\frac{8u^2}{H_s^2}}
\end{aligned}
$$

with $\mathbf{S} = [0, X] \times [0, Y] \times [0, T]$

Wave crest height and wave velocities

- $$\begin{aligned} P(\max_{\boldsymbol{\tau}\in\mathbf{S}} W(\boldsymbol{\tau}) \geq u) &\leq \left(\frac{X}{L_x} + \frac{Y}{L_y} + \frac{T}{T_z}\right) e^{-\frac{8u^2}{H_s^2}} \\ &+ \sqrt{2\pi}\left(N_t + N_y\sqrt{1-\alpha_{xt}^2} + N_x\sqrt{1-\alpha_{yt}^2}\right)\frac{4u}{H_s} e^{-\frac{8u^2}{H_s^2}} \\ &+ 2\pi N\sqrt{1-\alpha_{xt}^2-\alpha_{yt}^2}\,\frac{16u^2}{H_s^2} e^{-\frac{8u^2}{H_s^2}} \end{aligned}$$

  with $\mathbf{S} = [0,X]\times[0,Y]\times[0,T]$
- For $X = Y = 0$,

  $$P(\max_{t\in[0,T]} W(t) \geq u) \leq \frac{T}{T_Z} e^{-\frac{8u^2}{H_s^2}}$$

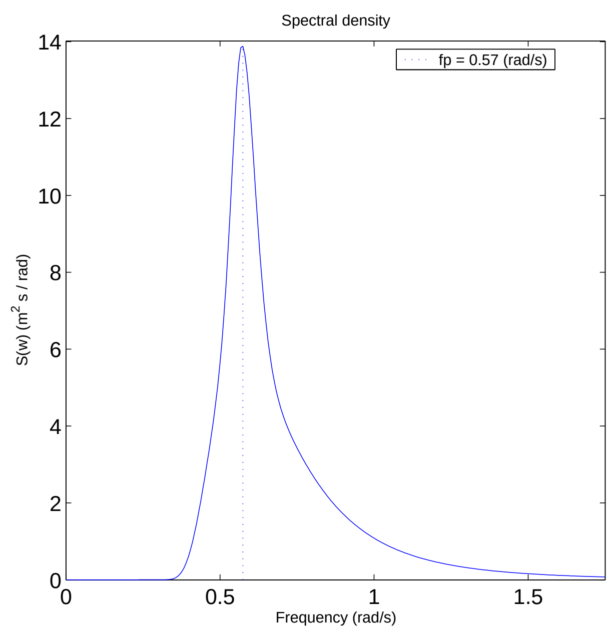
Wave characteristics for spreading functions.

Area: $100m \times 100m$ for 10 s

$D(\omega, \theta)$	L_x	L_y	T_z	α_{xt}	v_x
box	124.87	154.45	8.7949	0.9228	-13.1023
cos2s	120.54	164.22	8.7988	0.89	-12.1936
sech2	117.62	172.45	8.7988	0.8751	-11.699
misses	117.25	173.63	8.7988	0.8658	-11.5381
poisson	110.89	199.68	8.7882	0.8202	-10.3492

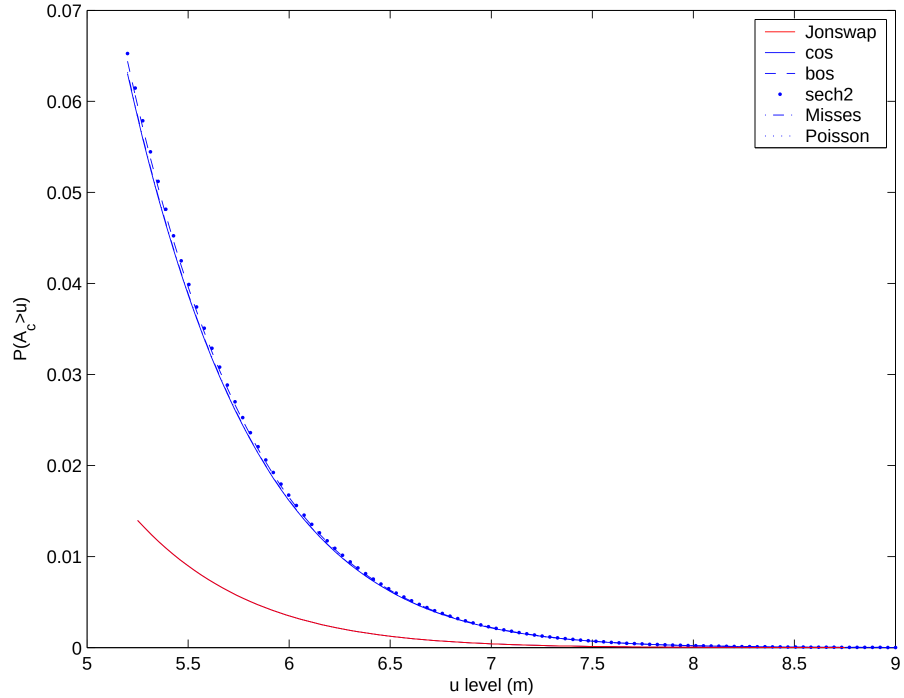
Linear Sea model - Gaussian field

- $S(\omega, \theta) = S(\omega) D(\omega, \theta)$
- $S(\omega)$ *frequency spectrum* usually JONSWAP with $H_s = 7m \quad T_p = 11s$ and $\gamma = 2.3853$
- $D(\omega, \theta)$, *spreading function*, like box, cos2s, sech2, Misses, Poisson

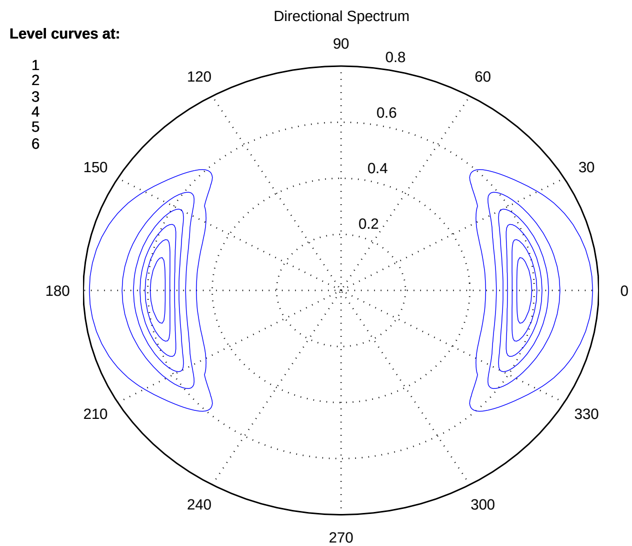


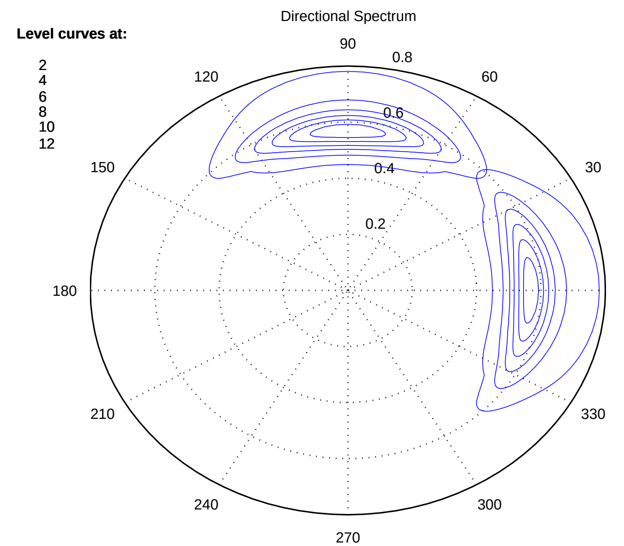
Wave Crest in a Gaussian Sea

$100m \times 100m \times 10s$



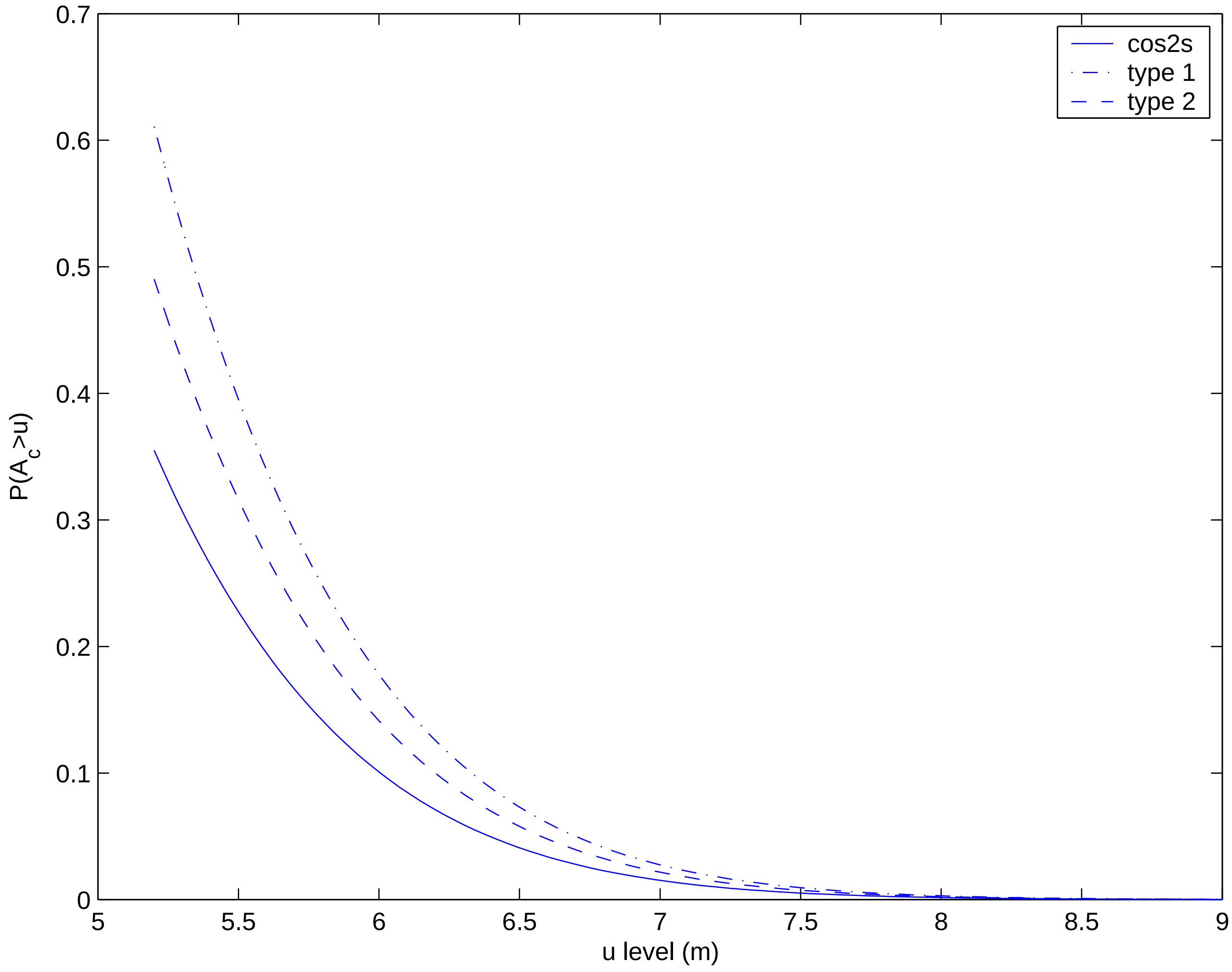
Sea spectra





spectra	L_x	L_y	α_{xt}	α_{yt}	v_x	v_y
cos2s	120.54	164.22	0.89	0	-12.19	0
type 1	120.54	164.22	0	0	0	0
type 2	137.42	137.42	0.51	0.51	-7.92	-7.92

Effect of directionality to distribution of high crests



Freak Waves

Freak wave: $A_c \geq 0.72 \times 2.1 H_s$

- Buoy: For $u = 1.5H_s \Rightarrow \frac{1}{\mu^+(u)} \approx 18.5$ years and if $T_z \approx 10s \Rightarrow 60$ mil. waves
- Area $5km \times 10km$, $P_{xy}(u) = 0.00058 \Rightarrow \frac{1}{P_{xy}(u)} \approx 1723$ pictures or ≈ 4.5 mil. waves
- Area $5km \times 10km$ for 37 min $\Rightarrow \frac{1}{P_{xyt}(u)} \approx 1 \Rightarrow \approx 640000$ waves

Conclusions

- Seas with different spreading functions resulted to the same global maximum distribution
- confused seas = highest waves (no organised movement)
- drifting seas (narrow-band) = waves moving along one direction with high speeds = lower extreme crest heights

Problem

Find h^{crt} so that

$$P(max_{0 \leq t \leq T} X(t) > h^{crt}) = p_0$$

- $X(t)$ - sea surface at time t as a second order sea
- $T = 1$ year
- $p_0 = 10^{-4}$, then h^{crt} is called the 10 000 year wave crest

Review of methods

When p_0 is small, h^{crt} takes large values $\Rightarrow$

- Statistical methods (i.e. Peaks Over Threshold (POT) method or yearly maxima) require crest height measurements over large periods of time

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When p_0 is small, h^{crt} takes large values $\Rightarrow$

- Statistical methods (i.e. Peaks Over Threshold (POT) method or yearly maxima) require crest height measurements over large periods of time
- Mathematical models: assume independence of wave crests, see Forristall (2000), Prevosto et al. (2000) and Krogstad and Barstow (2004)

Rice method

Let $N_T^+(h)$ be the number of upcrossings of the level h by the process $X(t)$ during the time period $[0, T]$.
Then for any fixed time $t_0 \in [0, T]$

- $P(\max_{0 \leq t \leq T} X(t) > h) \leq \mathsf{E}[N_T^+(h)]$

Rice method

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Then for any fixed time $t_0 \in [0, T]$

- $P(\max_{0 \le t \le T} X(t) > h) \le \mathsf{E}[N_T^+(h)]$
- Problem is equivalent to computing $\mathsf{E}[N_T^+(h)]$

Computation of $\mathsf{E}[N_T^+(h)]$

$$\mathsf{E}[N_T^+(h)] = \int_0^T \mu_t^+(h)\,dt = \int_0^T \int_0^{+\infty} z f_{X(t),\dot{X}(t)}(h,z)\,dz\,dt$$

- The density $f_{X(t),\dot{X}(t)}(h,z)$ includes two sources of variability
  - Variable sizes of sea waves during a sea state $S_t := S_t(\omega,\theta)$ for fixed t
  - Evolution of sea states with t

Computation of $\mathsf{E}[N_T^+(h)]$

S_t: random sequence of sea states. Then

$$\int_0^{+\infty} z f_{X(t),\dot{X}(t)}(h,z)\,dz = \mathsf{E}[\int_0^{+\infty} z f_{X(t),\dot{X}(t)|S_t}(h,z)\,dz]$$

- Change rate of $\{S_t\}$ much slower than of sea elevation, so we approximate $f_{X(t),\dot{X}(t)|S_t}(h,z)$ by that of a second order sea with S_t

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- Change rate of $\{S_t\}$ much slower than of sea elevation, so we approximate $f_{X(t),\dot{X}(t)|S_t}(h,z)$ by that of a second order sea with S_t
- $\int_0^{+\infty} z f_{X(t),\dot{X}(t)|S_t}(h,z)\,dz \approx \mu^+(h|S_t)$

Estimation of $\mu^+(h|S)$

In the case of a stationary process and for fixed t

$$\mu^+(h|S_t) = \int_0^{+\infty} z f_{X(t),\dot{X}(t)|S_t}(h,z)\,dz,$$

where $X(t)$ is a second order sea

- saddle point method
- Breitung method: $\mu^+(h|S) \approx c(\beta_h)\exp(-\beta_h^2/2)$ where β_h is the Hasofer-Linds safety index

Evaluation of $E[N_T^+(h)]$

- $\{S_t\} \equiv (H_s(t), T_z(t)) \Rightarrow \mu^+(h|S_t) = \mu^+(h|H_s(t), T_z(t))$

Long term distribution means we look at a sea state chosen at random

Evaluation of $E[N_T^+(h)]$

- $\{S_t\} \equiv (H_s(t), T_z(t)) \Rightarrow \mu^+(h|S_t) = \mu^+(h|H_s(t), T_z(t))$
- The long run distribution of $(H_s(t), T_z(t))$ exists
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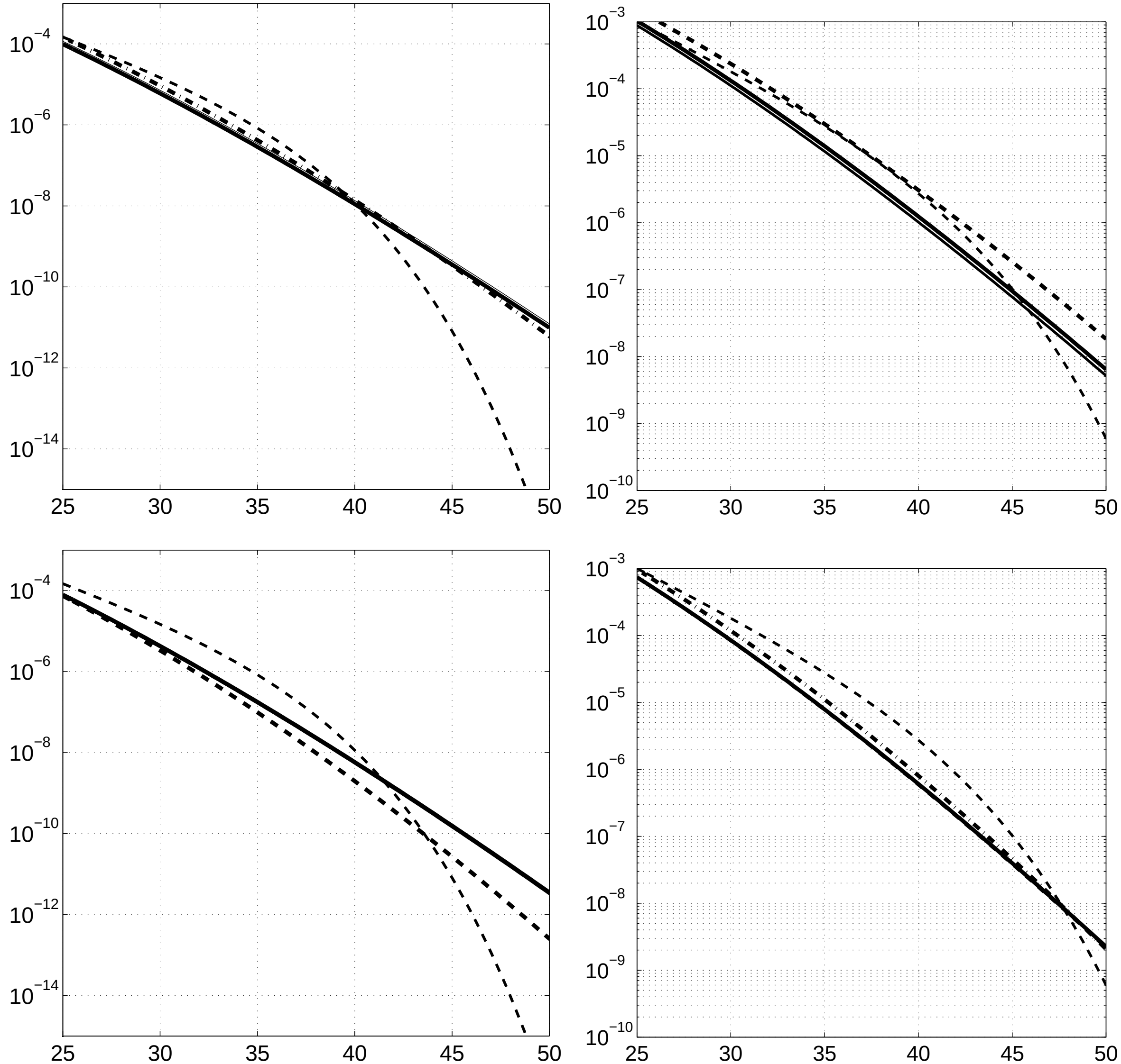
  Hence
- $E[N_T^+(h)] \approx T\mu^+(h)$

Long term distribution means we look at a sea state chosen at random

Comparison of short term distribution



Solid: FORM and SORM, light dashed: Dawson, thick dashed: Forristall model

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- Forristall, G. Z., (2000). Wave crest distributions: Observations and second order theory, *Journal of Physical Oceanography*, Vol. 30, pp. 329-360.
- Krogstad, H. E., and Barstow, S., (2004). Analysis and applications of second-order models for maximum crest height, *Journal of Offshore Meachanics and Arctic Engineering*, Vol. 126, pp. 66-71.

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Breitung method

Let $g : \mathbb{R}^n \to \mathbb{R}$ be s.t. $S = \{\mathbf{x} = (x_1, \ldots, x_n); g(\mathbf{x}) = 0\}$ has a unique point $\mathbf{x}_0$ of minimum distance from the origin. Suppose $\mathbf{Z}(t)$ is an n-dimensional, stationary, differentiable, Gaussian vector process, and $\dot{\mathbf{Z}}(t)$ its derivative. Then for $g(\mathbf{Z}(t)/\beta)$, $\beta > 0$, under some mild technical assumptions, we have:

$$\mu_\beta^+(0) = \frac{\mathrm{e}^{-\beta^2/2}}{2\pi}(c+O(\beta^{-2})), \qquad c = \sqrt{\frac{\mathbf{x}_0^T\left(\Sigma_{22} - \Sigma_{21}G_0\Sigma_{12}\right)\mathbf{x}_0}{\det\left(I + P_0G_0P_0\right)}}$$

as $\beta \to \infty$ where $G_0 := \frac{1}{|\nabla g(\mathbf{x}_0)|}\left[\frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x}_0)\right]_{i,j=1,2,\ldots,n}$

$P_0 := I - \mathbf{x}_0\mathbf{x}_0^T$ and Σ is the covariance matrix of $(\mathbf{Z}(t),\dot{\mathbf{Z}}(t))$.