Estimating Extremes

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SEAMOCS, Palmse, Oct 2007

Extremes of wave characteristics from data?   How uncertain?

1. Data, classical extremal estimation,
   uncertainties...
2. Refinements and Complications
3. Developments

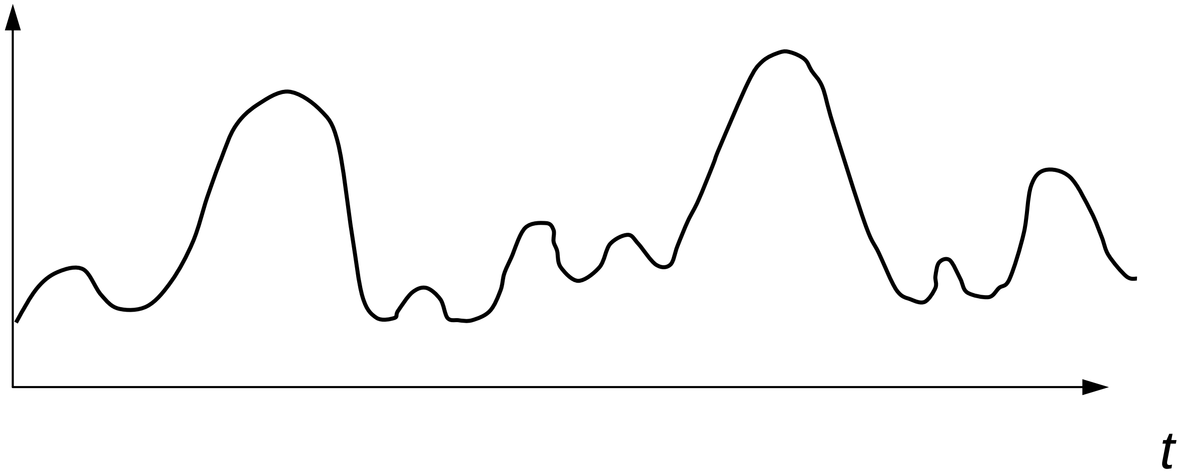
Specific focus: Estimation of high quantiles, return levels

1 Data, Classical Methods

Data:

- buoy, platform
- model
- satellite
- ship

} usually regular, frequent



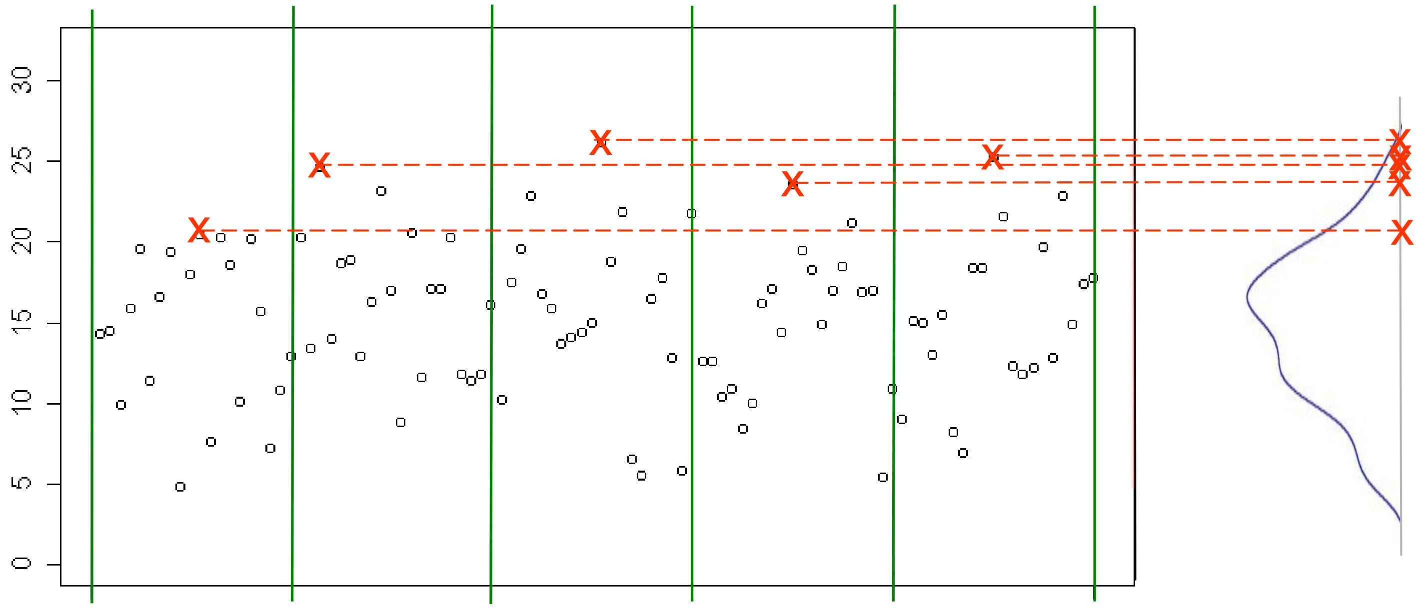
General Strategy: allow extreme observations to drive estimation model

'Nobody believes in a ~~theory~~ except the person who invented it; everyone believes in an observation except the person who made it.'

Albert Einstein

(a) Classical Estimation: Block Maxima

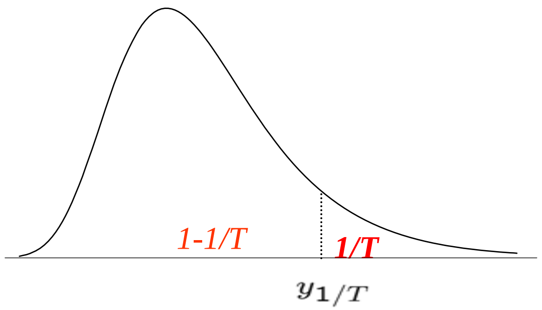
usually annual maxima



Fit a Generalized Extreme Value distribution to observed maxima

$$G(x) = \exp\left\{-\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]_+^{-1/\xi}\right\}$$

giving the $1 - 1/T$ quantile

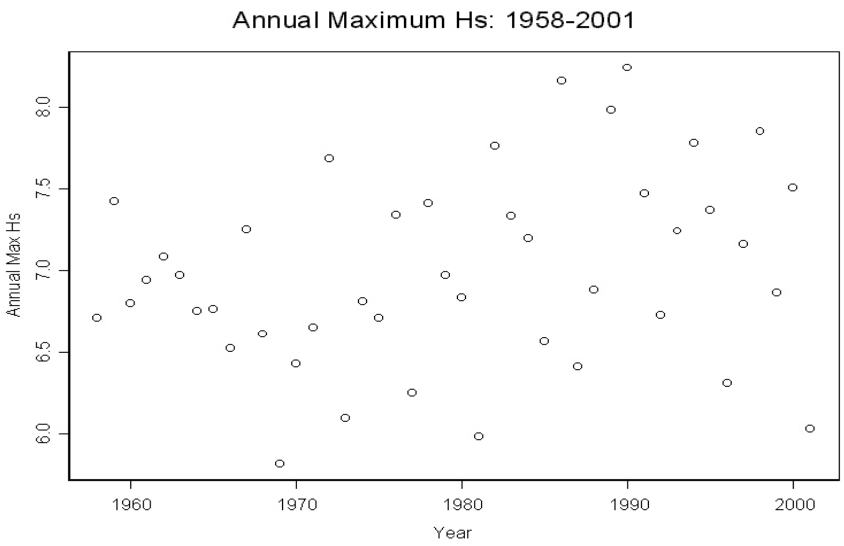


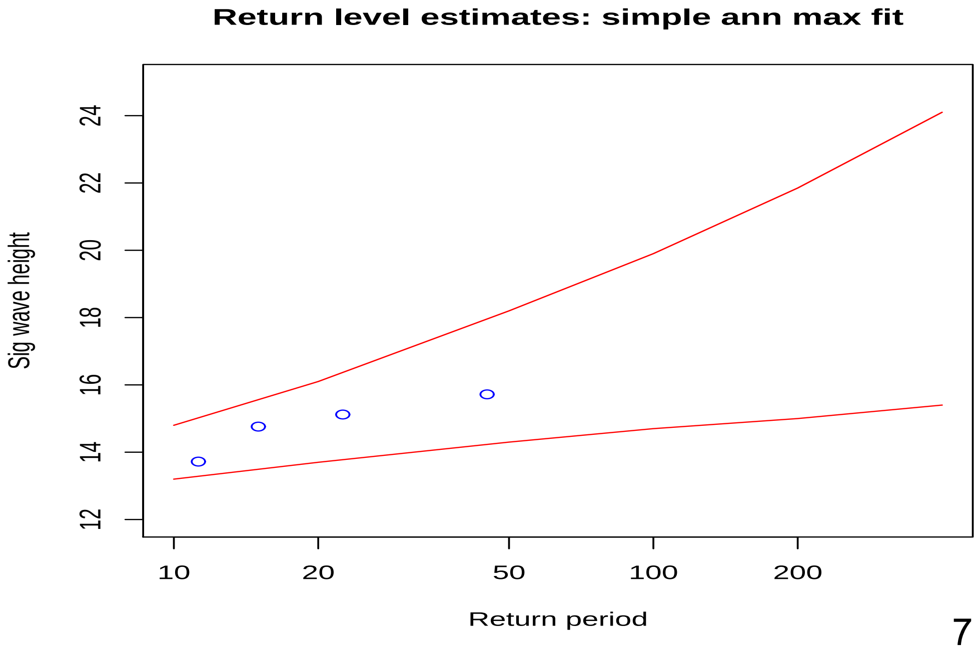
T-year return level

$$r_T = \mu - \frac{\sigma}{\xi}\left[1 - \{-\log(1 - 1/T)\}^{-\xi}\right]$$

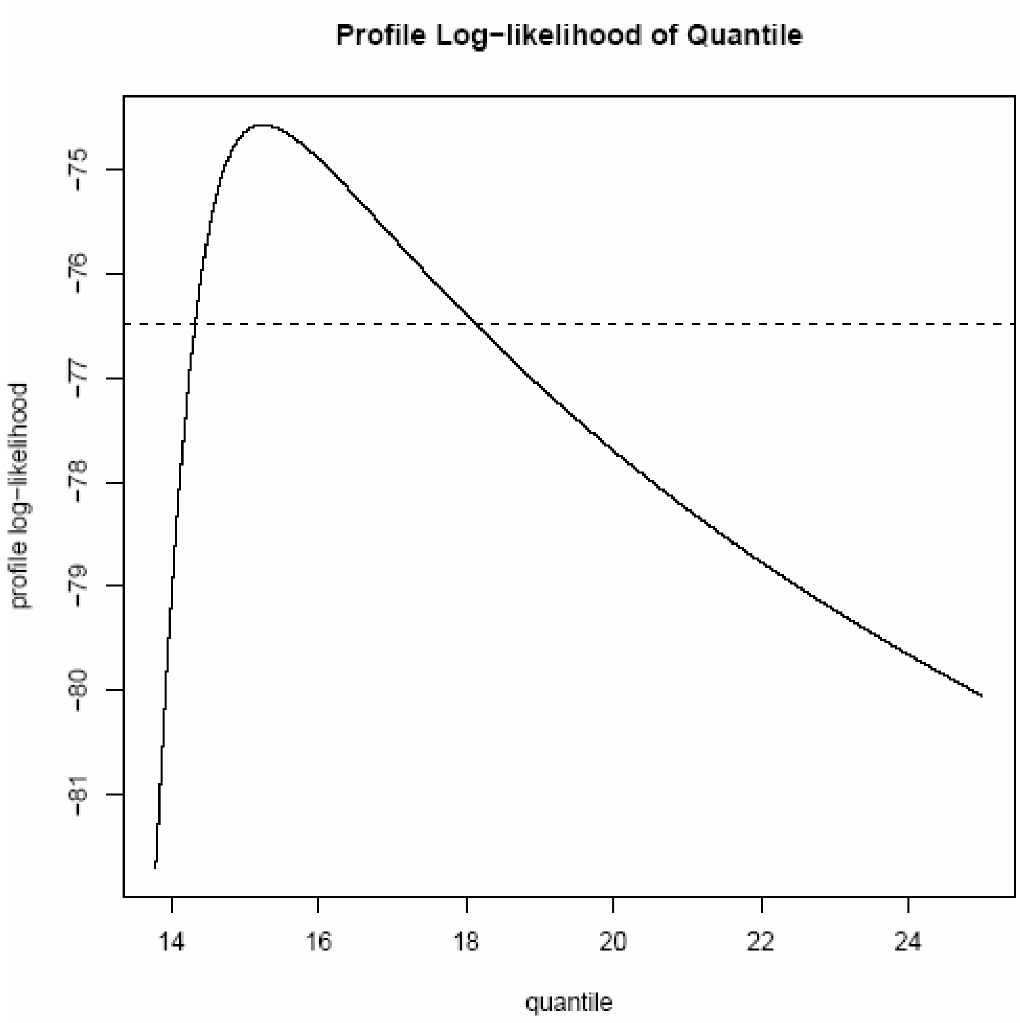
Maximum likelihood estimation of μ, σ, ξ leads to confidence intervals for r_T

ERA40 model data, 21°W 54°N

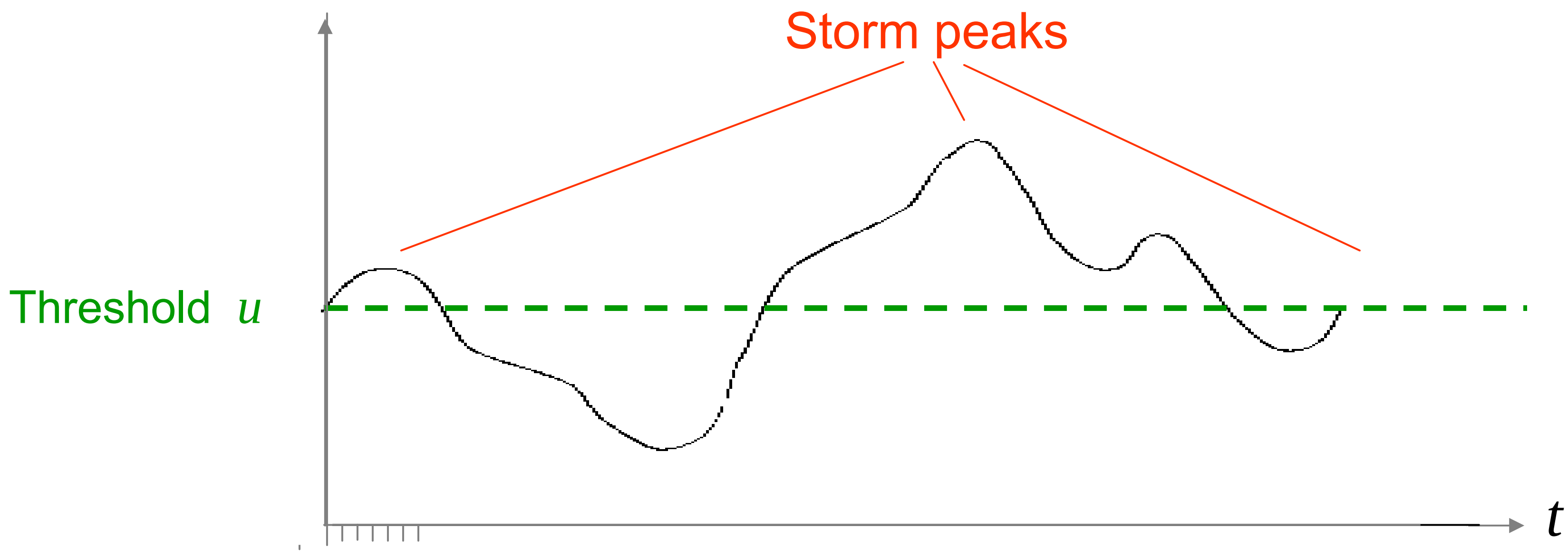




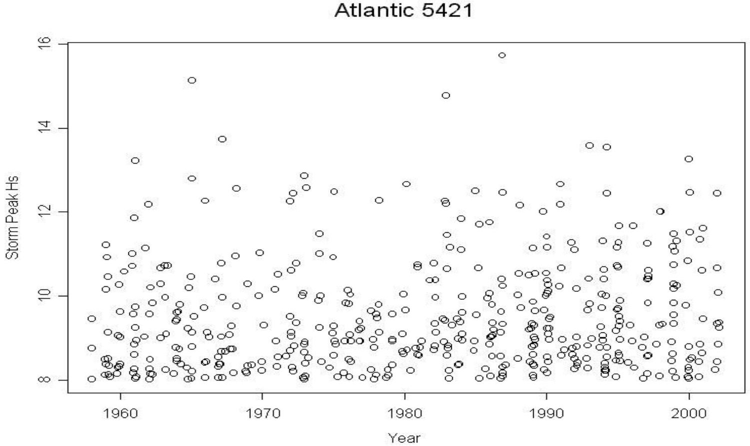
Example: Profile log-likelihood of 50-year return level



(b) Classical Estimation: Threshold Methods

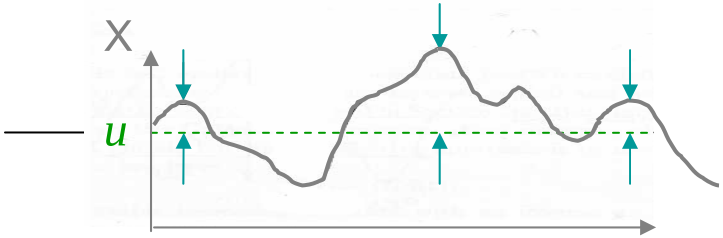


Concentrate attention on all storm peaks over a high threshold



Generalized Pareto distribution for

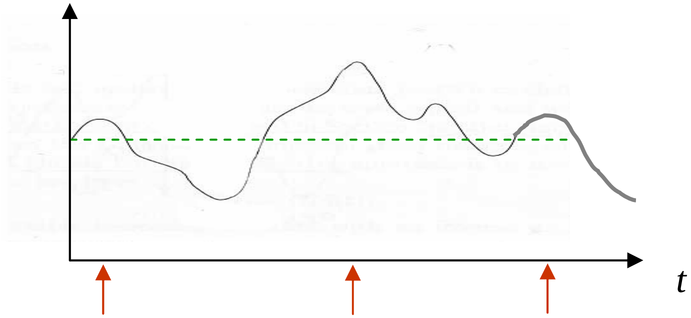
excesses of threshold u



$$Pr(X > u + y \,|\, X > u) = \left(1 + \frac{\xi y}{\sigma}\right)^{-1/\xi}, \quad y > 0$$

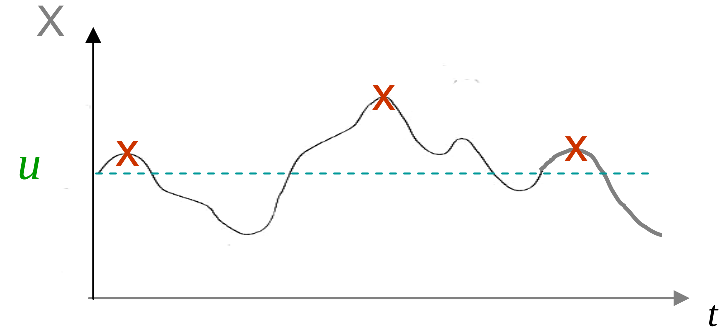
Poisson process of occurrence of storms

rate $\lambda(t)$



Point-process modelling:

a re-formulation, often more convenient for handling covariate dependence



Model $(t, x) =$ (times, sizes) of storm peaks by an inhomogeneous Poisson process with intensity density

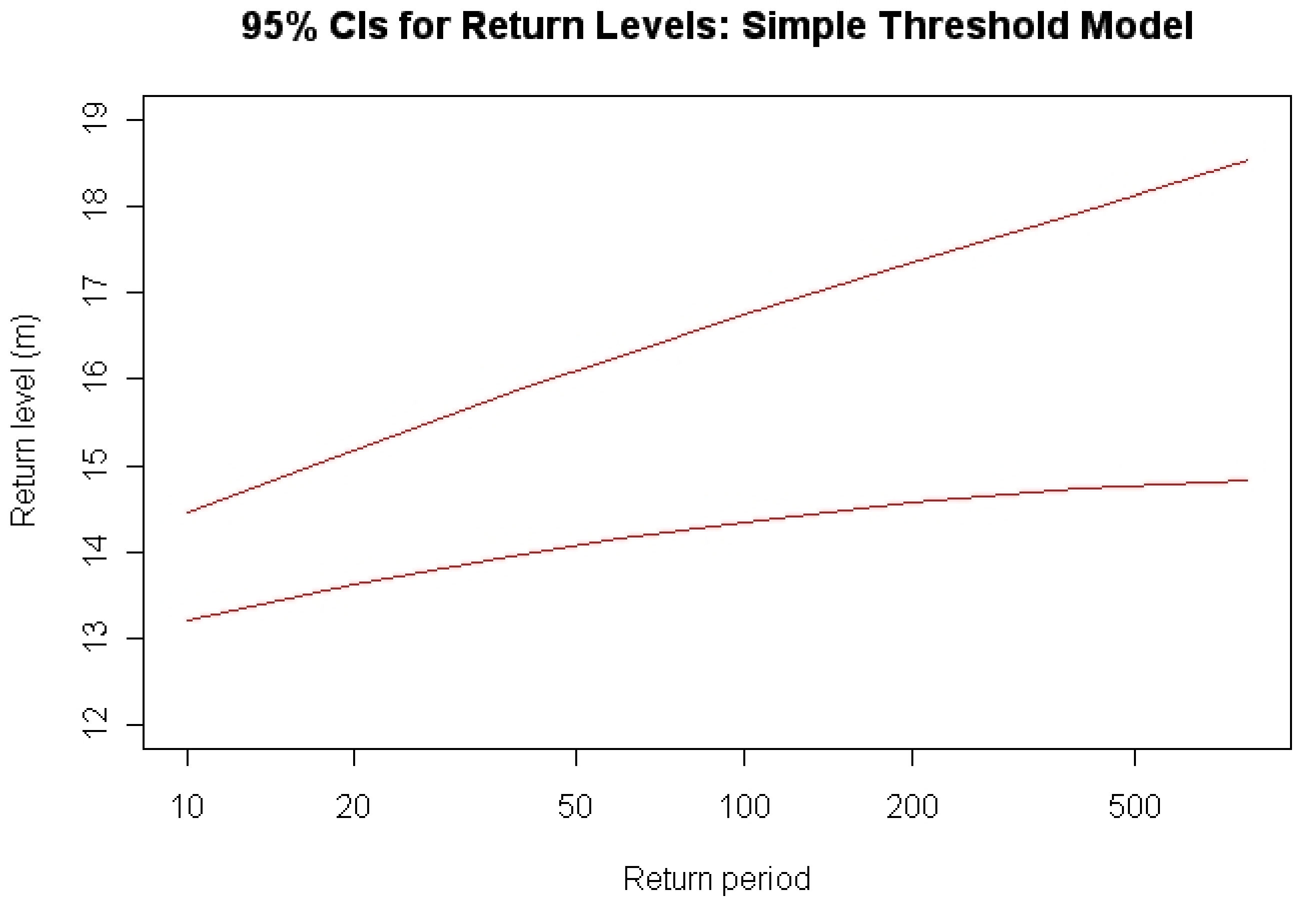
$$\lambda(t,x) = \frac{1}{\sigma_p(t)} \left\{ 1 + \xi_p(t) \frac{x - \mu_p(t)}{\sigma_p(t)} \right\}^{(-1/\xi_p(t))-1} \qquad ((t,x) \in C)$$

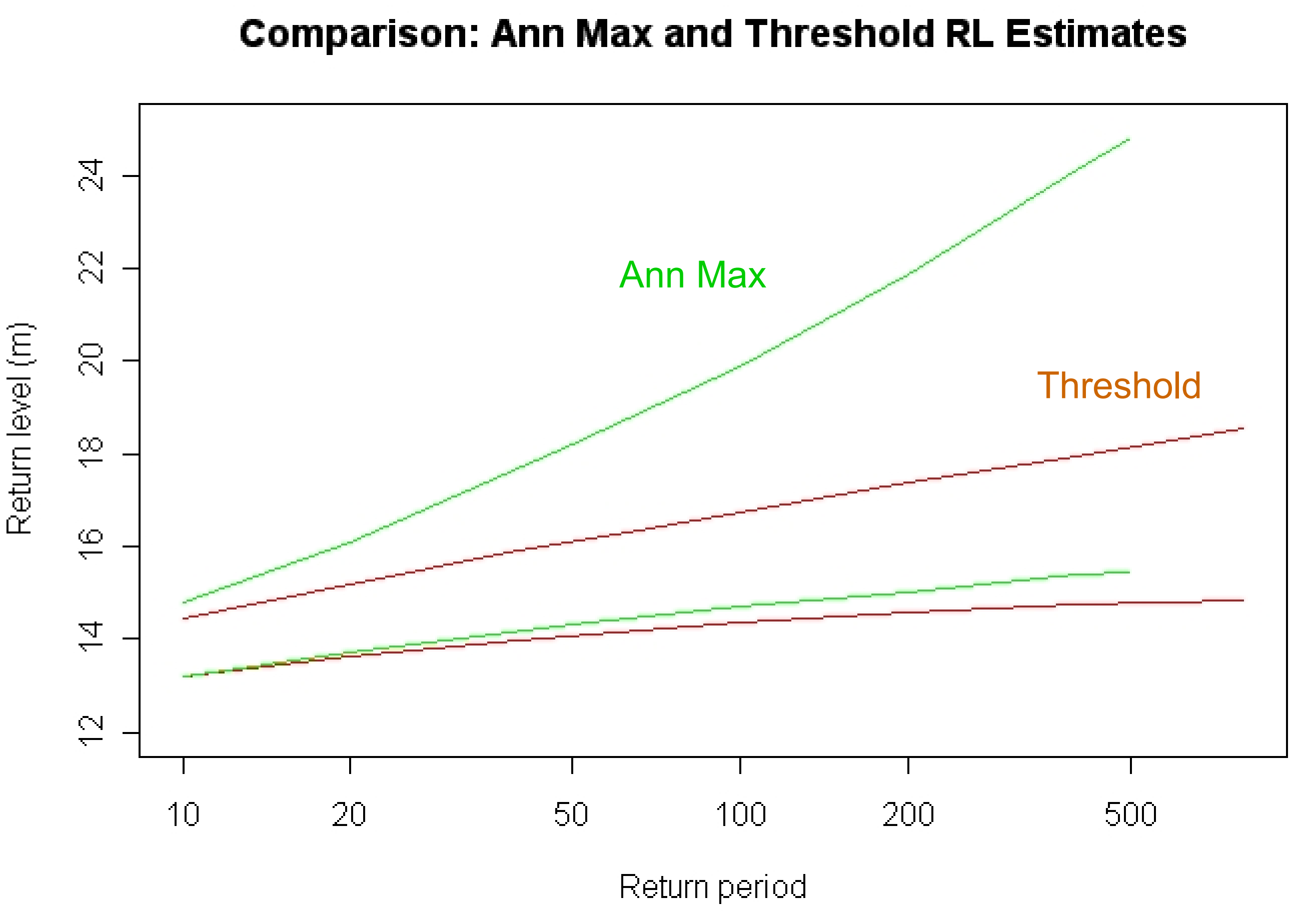
for a suitable region C, and possibly time-varying parameters σ_p, μ_p and ξ_p

Convenient for time-varying thresholds $u(t)$

Inference via likelihood methods as before.

ERA40 model data, 21°W 54°N



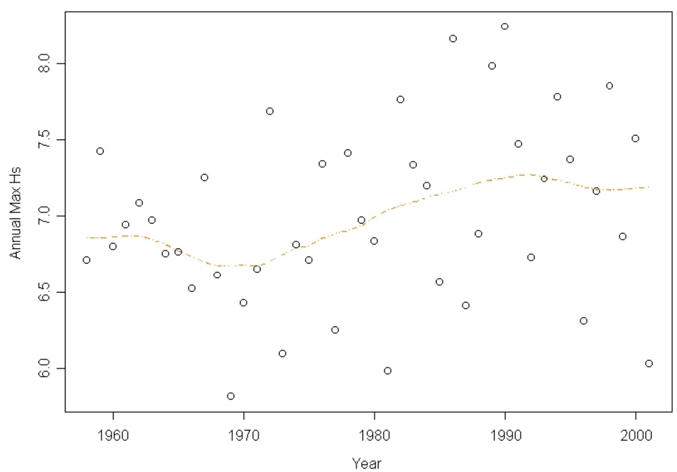


Comparison: Ann Max and Threshold RL Estimates
Ann Max
Threshold
Return level (m)
12
14
16
18
20
22
24
10
20
50
100
200
500
Return period

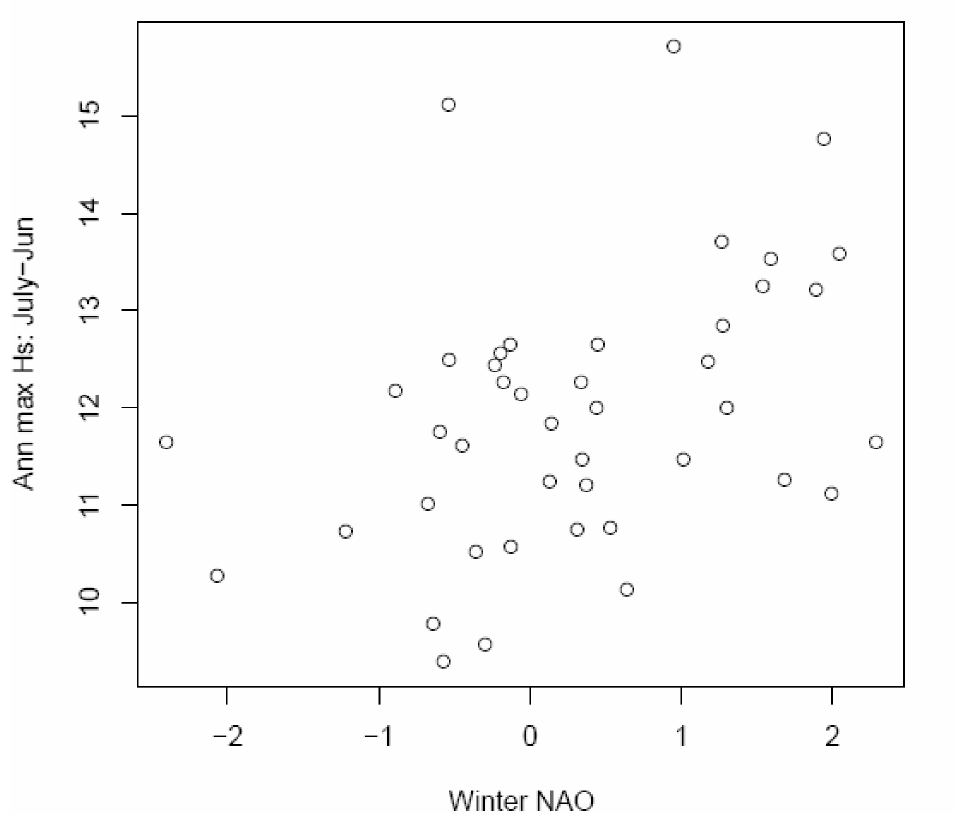
2 Refinements and Complications

(a) Non-stationarity, Covariate Dependence

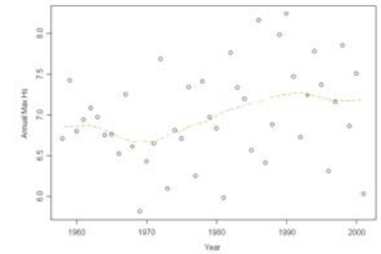
eg annual maximum possibly changing with time, with NAO, ...



Atlantic 5421 Annual Max (July-June) vs Winter NAO
1958/9 – 2001/2



Accommodate changes with time or in response to other variables by allowing GEV/GPD/point process parameters to vary.



eg $$\mu(t) = \beta_0 + \beta_1 z_1(t) + \ldots + \beta_p z_p(t)$$

known functions of t

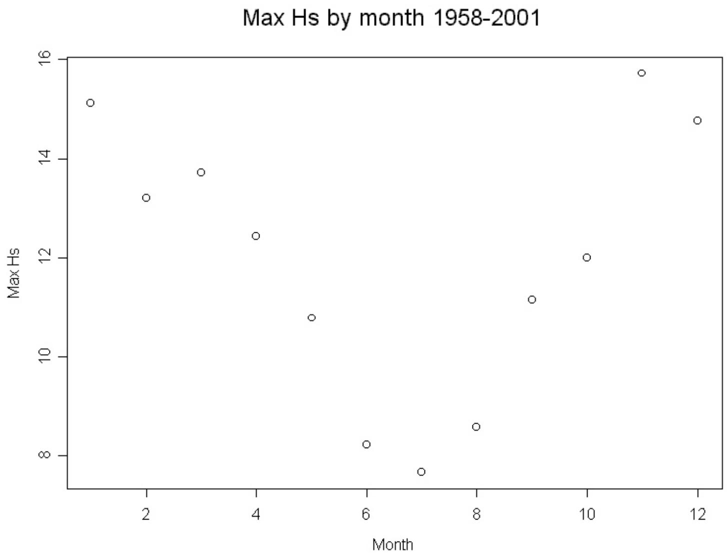
for new parameters $\beta_0, \beta_1, \ldots$ which can be estimated and tested.

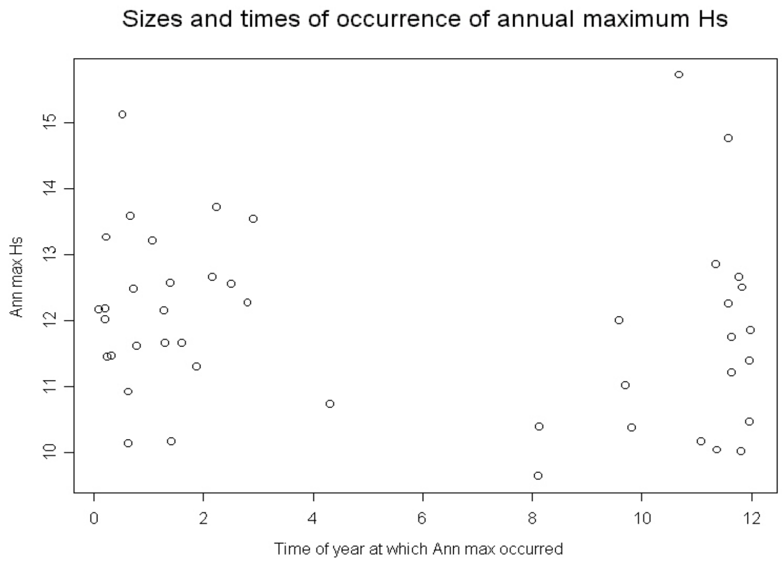
eg $$\beta_0 + \beta_1 t + \beta_2 \cos(2\pi t) + \beta_3 \sin(2\pi t) + \beta_4 NAO(t)$$

and similarly for ξ and $\log \sigma$

Atlantic 5421 data: strong evidence of NAO association,
no evidence of linear time-trend

(b) Seasonality



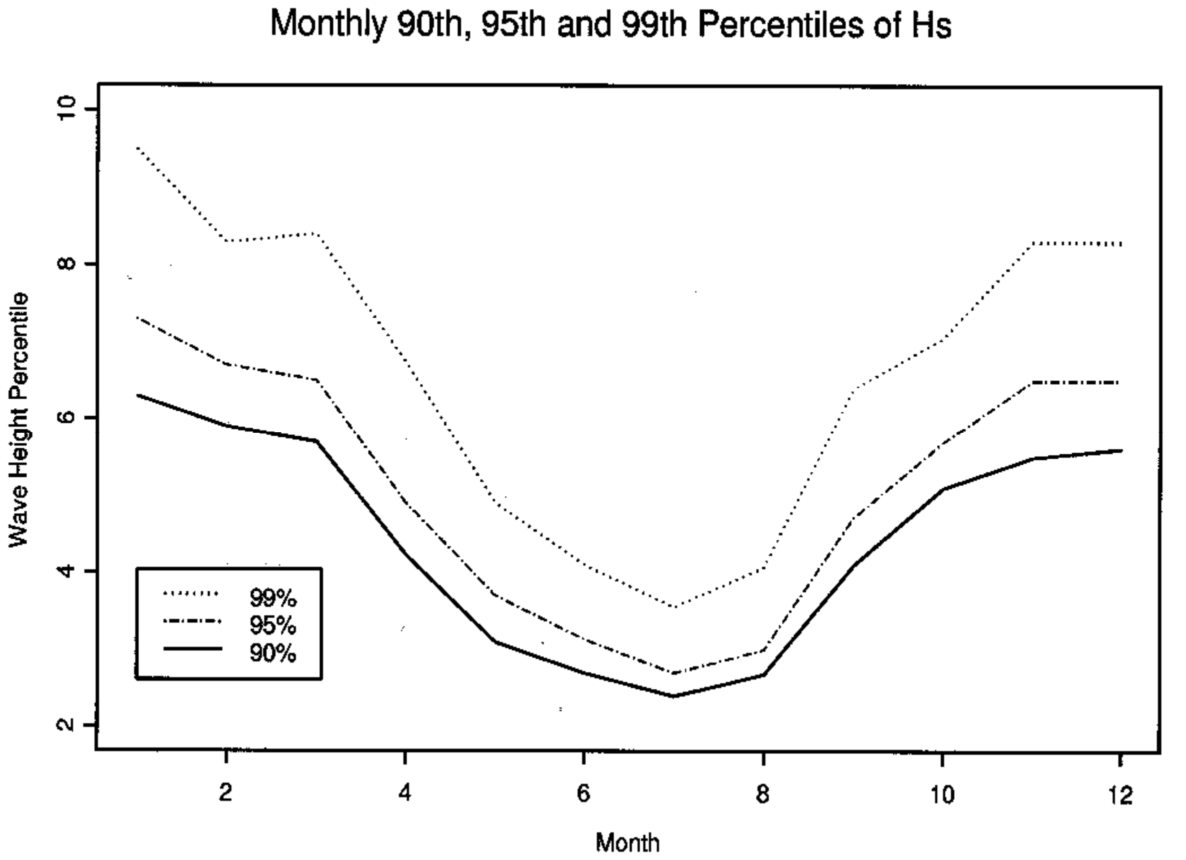


Take account of seasonality in estimation

– by using seasonally-varying parameters in either block maxima or threshold modelling

Wave heights over 5m, Northern N. Sea 1979-1999

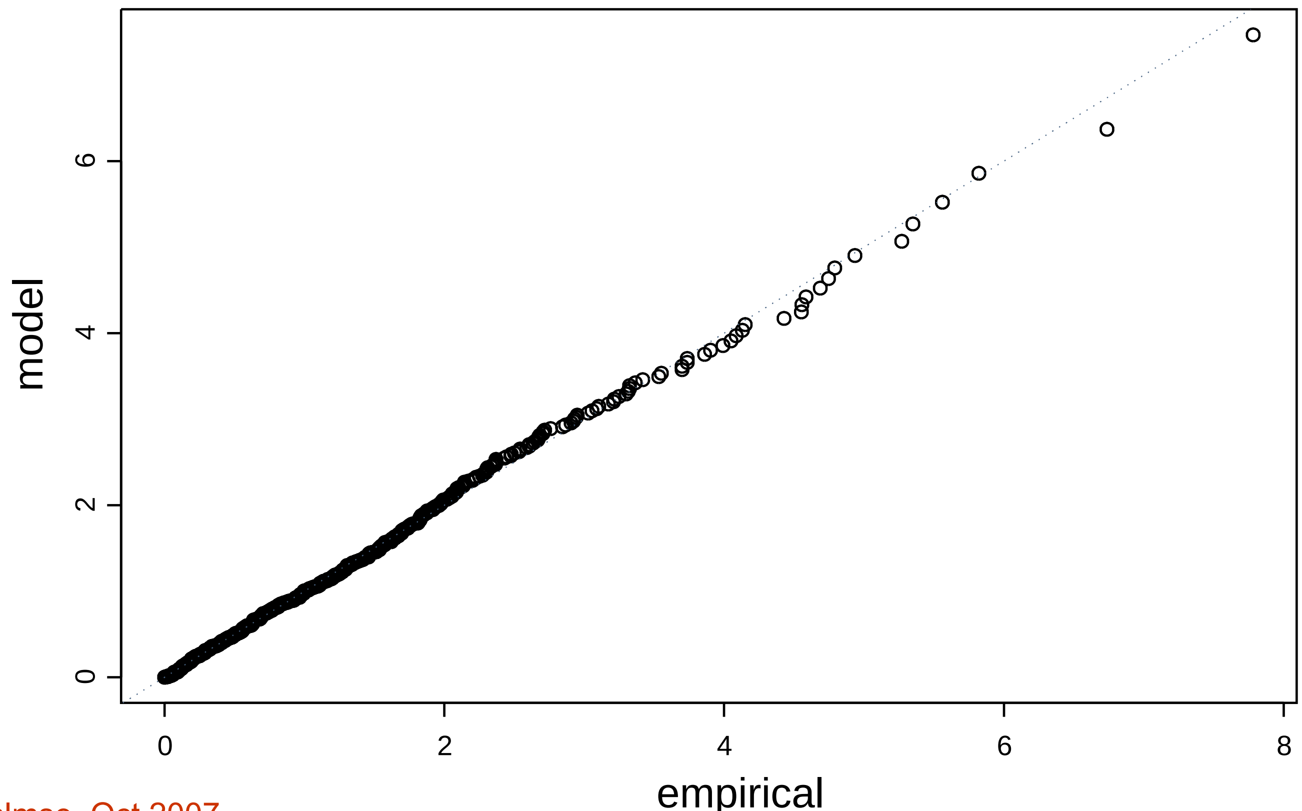
Seasonality



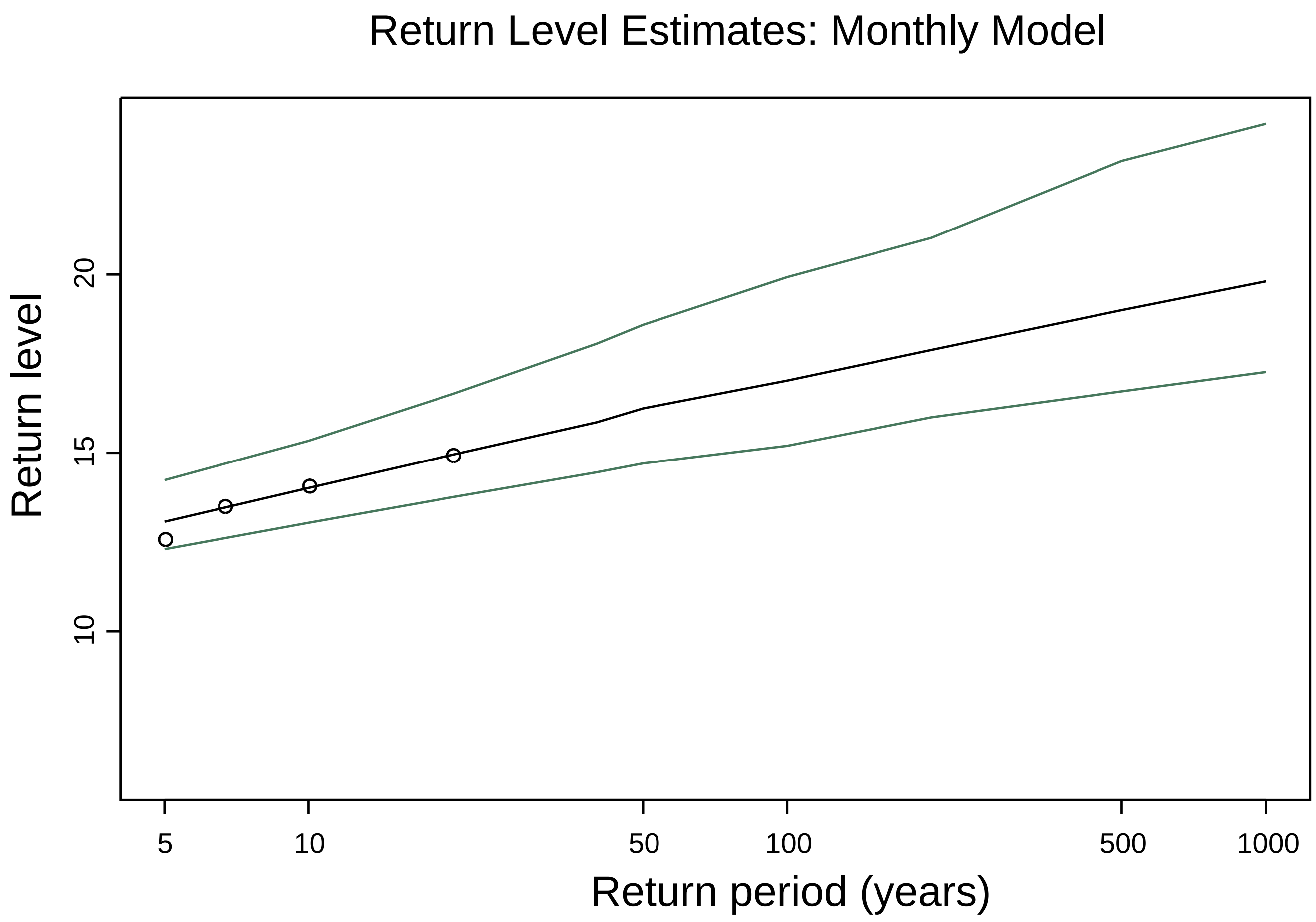
Separate-Months Point Process Model, N North Sea

$$
\begin{aligned}
\mu_p(t) &= \alpha_i \quad \text{for} \quad t \in \quad \text{Month}_i \\
\log(\sigma_p(t)) &= \beta_i \quad \text{for} \quad t \in \quad \text{Month}_i \\
\xi_p(t) &= \xi, \quad \text{for all } t,
\end{aligned}
$$

Standardized QQ Plot: Monthly Model



95% Confidence Intervals for Return Levels, NN Sea



The Effect of Neglecting Seasonality

(Carter & Challenor (1981))

100 year return level, Northern N Sea

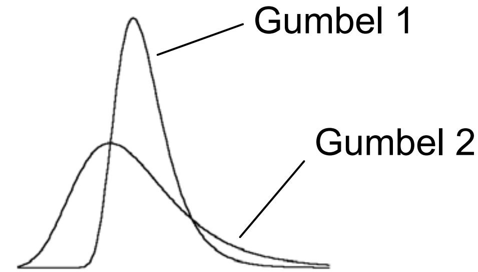
Model	ML Estimate	95% CI
Stationary	15.2	(13.8, 16.6)
Seasonal: separate months	17.0	(15.2, 19.9)

... under-estimation, over-confidence in this case

Explanation - 2-season case

Annual max = max of Season 1 and Season 2

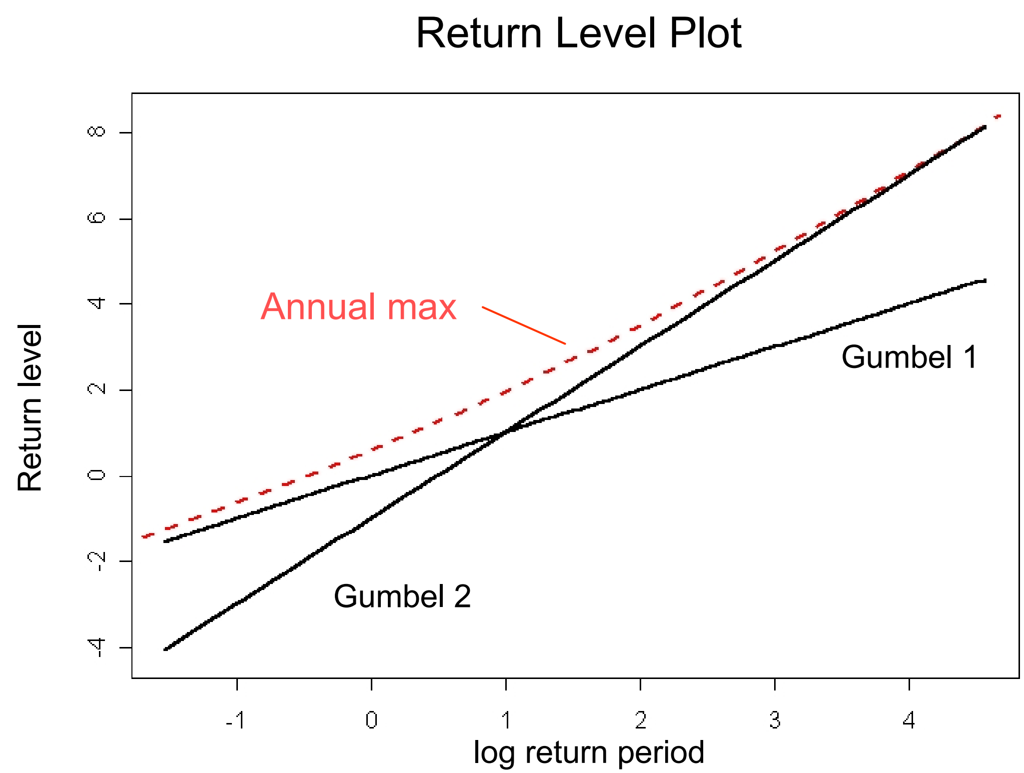
suppose independent Gumbel distributed



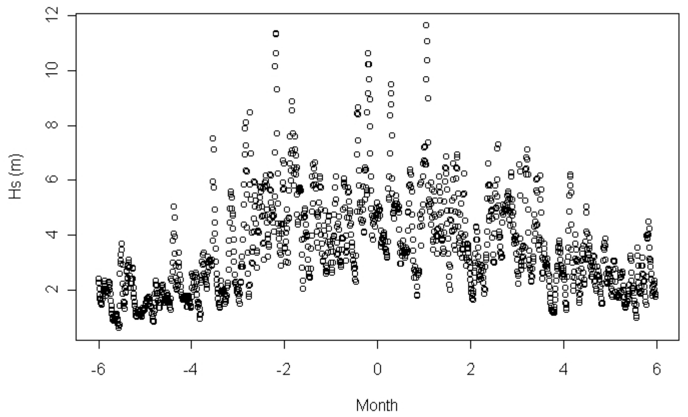
A single Gumbel fitted
to Annual Max
≡ linear approximation to

so underestimates true return
levels at high return periods

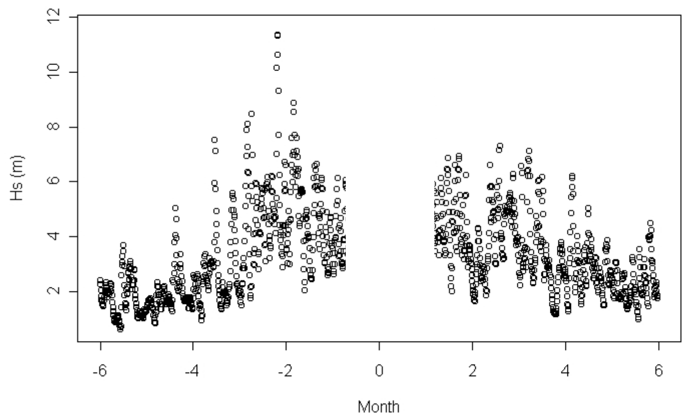
Moreover se of ret level est
is $\propto$ estimated slope, so
underestimates true uncertainty
too



(c) Effect of gaps in data



Annual maxima data: analysis based on censoring possible, but...

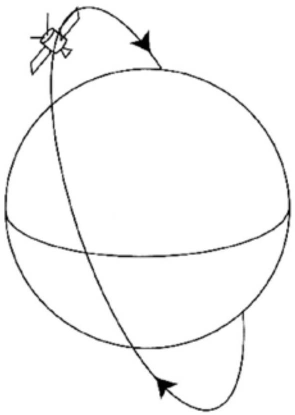


Threshold method: effect easier to accommodate, likely to have smaller influence

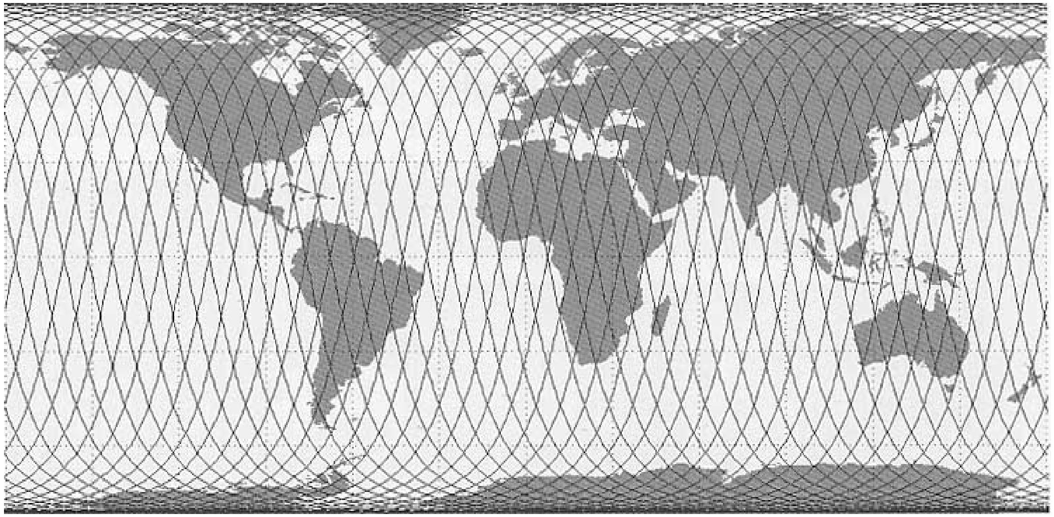
In general:

- loss of precision
- if data missing because of size, more detailed statistical modelling needed

(d) Estimation from Satellite Data



Ground track: ERS-1, 3 day repeat

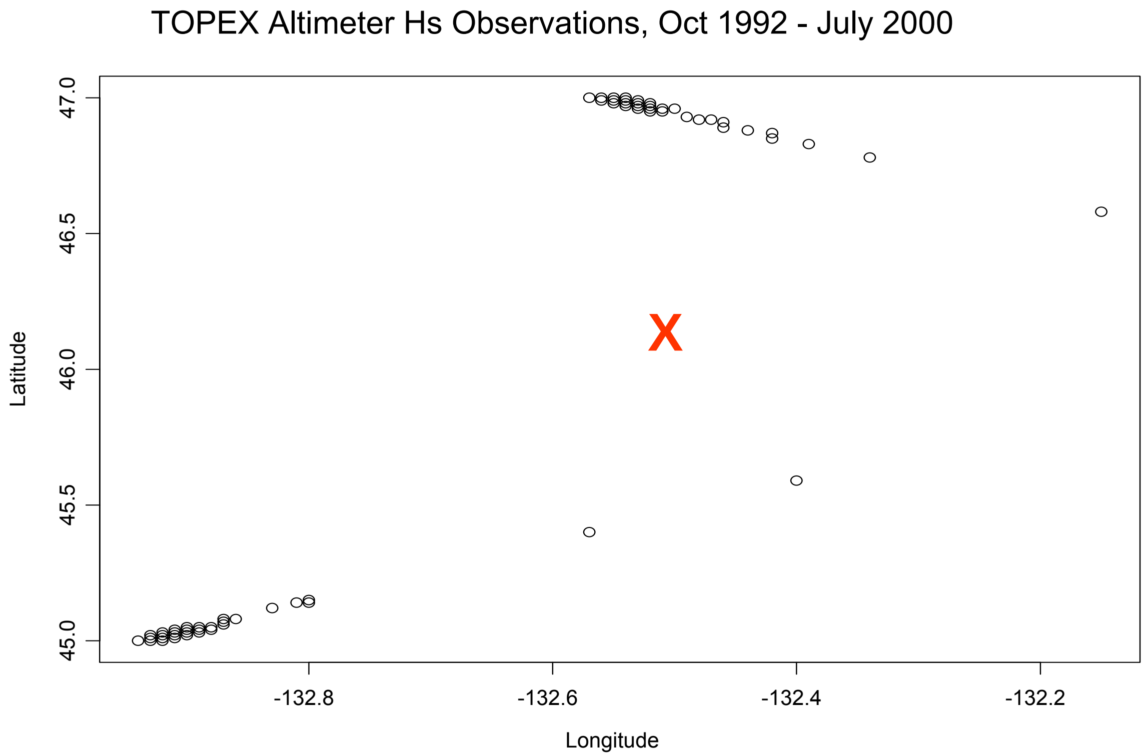


Problems:

- Spatial — no data from location of interest
- Temporal — miss storm peaks

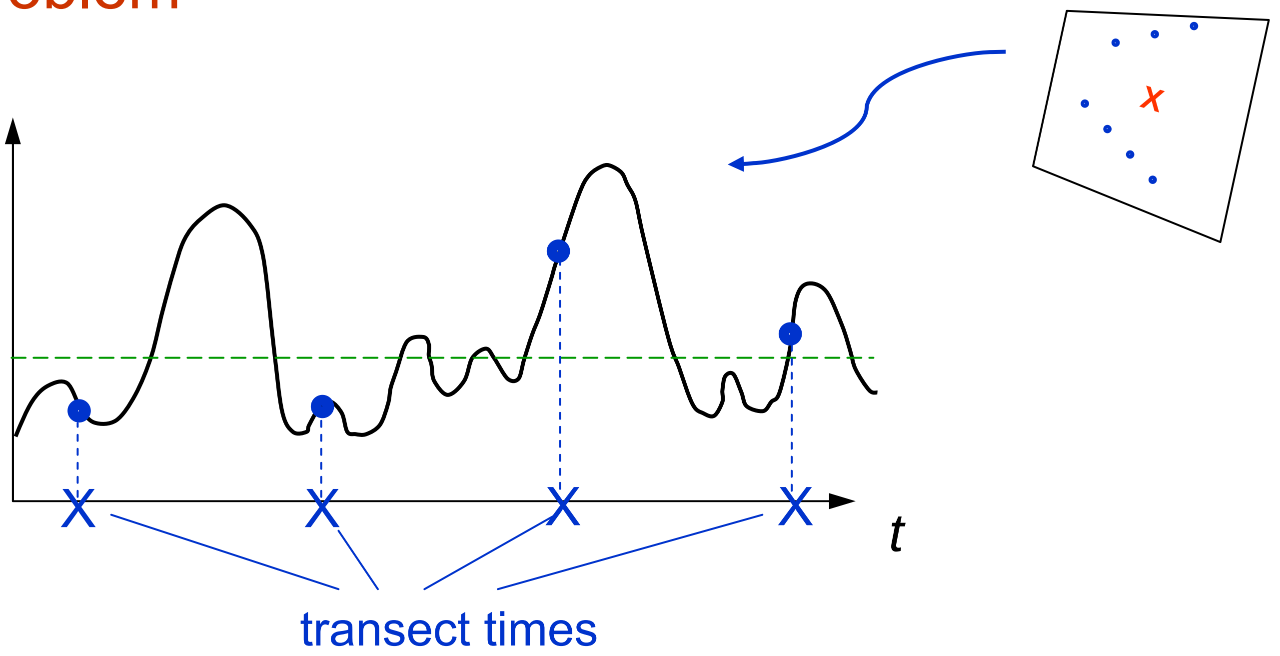
Spatial problem

Example: Wave heights off Vancouver
TOPEX observations



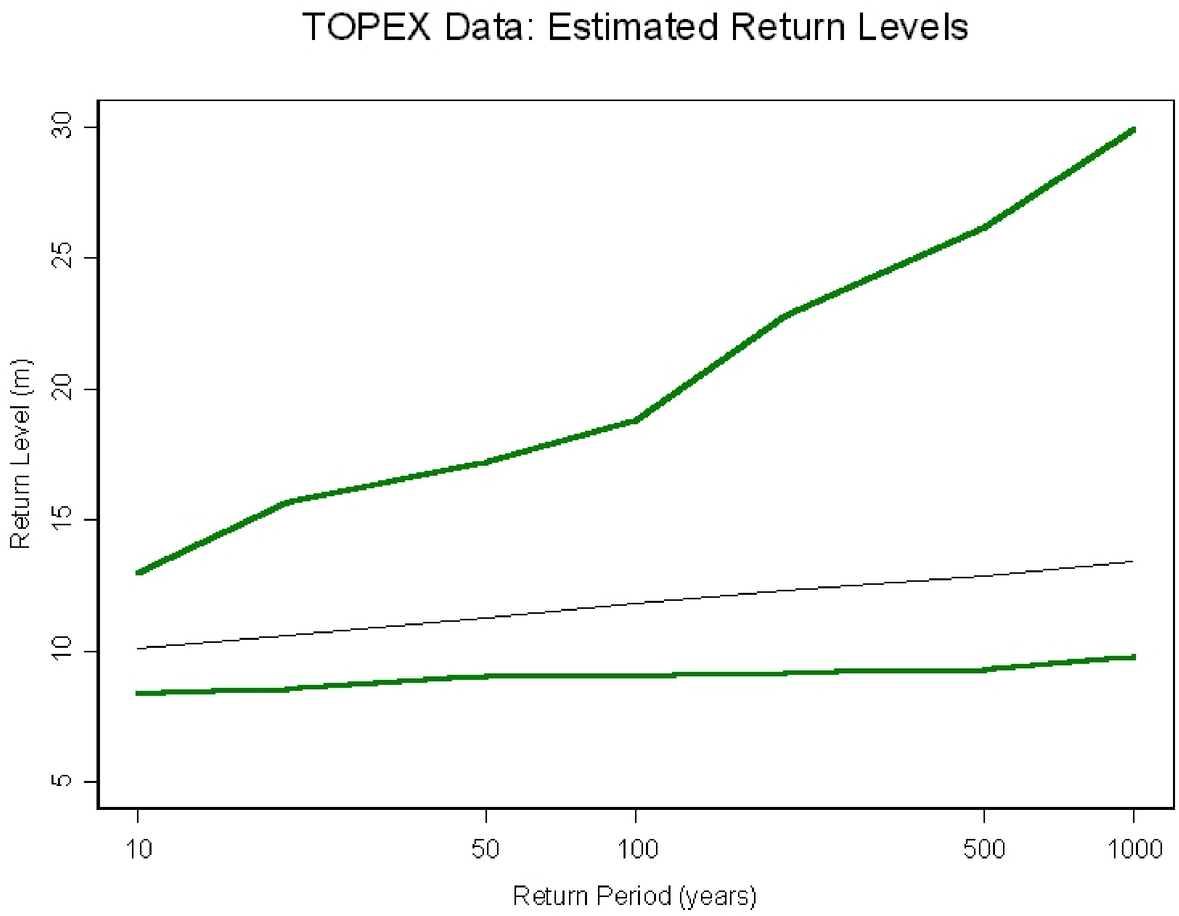
Take nearby observations to be representative of those at the target

Temporal problem



a) over-threshold observations unlikely to be storm peaks

b) many storms likely to be missed



eg 95% conf interval for 100-year return level for H_s = 9.1 – 18.8 m. No evidence of a trend in extremes

(e) Estimation from Multiple Sources of Data

eg satellite + buoy; data $\boldsymbol{x}_s$ and $\boldsymbol{x}_b$ say on extremes

Strategies:

i. Consistency checks over range of data

ii. For extremes: suppose true storm peak wave heights $\boldsymbol{x}$ are governed by a distribution $[\boldsymbol{x} \mid \boldsymbol{\theta}]$

and conditionally on $\boldsymbol{x}$ suppose observational models for data:

$$[\boldsymbol{x}_s \mid \boldsymbol{x}, \boldsymbol{\alpha}_s]$$

$$[\boldsymbol{x}_b \mid \boldsymbol{x}, \boldsymbol{\alpha}_b]$$

where $\boldsymbol{\alpha}_s, \boldsymbol{\alpha}_b$ are parameters specific to the data sources and $\boldsymbol{\theta}$ describes the underlying wave characteristics of interest

(eg $\boldsymbol{x}_s = \alpha_{0s} + \alpha_{1s}\boldsymbol{x} + \epsilon_s$

observation with bias and instrument error)

Then, if observation processes conditionally independent given $\boldsymbol{x}$,

likelihood is

$$L = \int_{\boldsymbol{x}} [\boldsymbol{x}_s \mid \boldsymbol{x}, \boldsymbol{\alpha}_s][\boldsymbol{x}_b \mid \boldsymbol{x}, \boldsymbol{\alpha}_b][\boldsymbol{x} \mid \boldsymbol{\theta}]$$

whence estimation and uncertainty estimates

If observation processes are dependent, then L is a pseudo-likelihood and inference still feasible but more challenging.

3 Developments

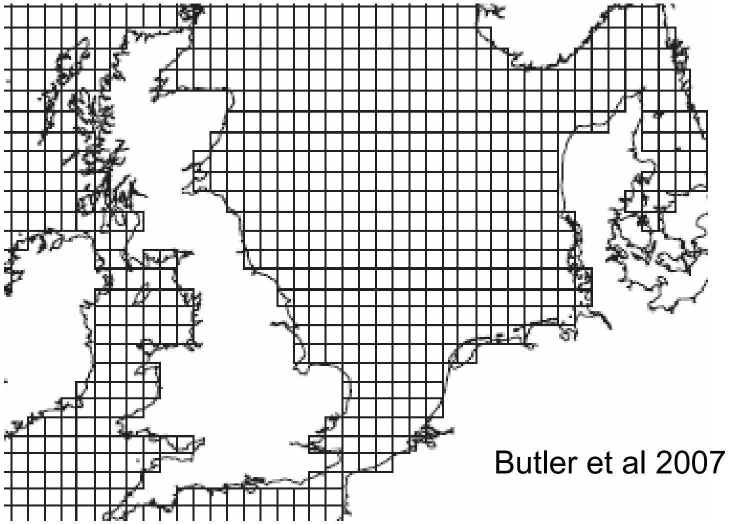
(a) Spatial Estimation

- neighbouring locations contain useful information about a site
- spatial structure is of interest in its own right

Represent spatial characteristics of extremes by allowing the parameters of the extremal distributions of §1 to depend smoothly (and non-parametrically) on location. Fit by local likelihood. Extends to spatio-temporal estimation

Chavez-Demoulin & Davison (2005), Butler et al (2007)

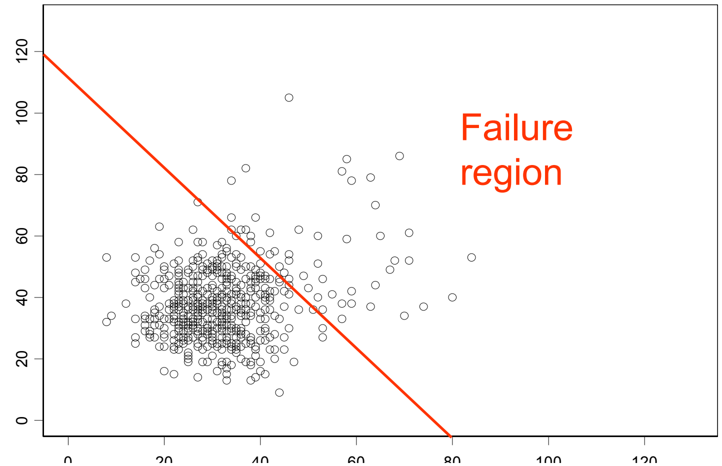
eg



Butler et al 2007

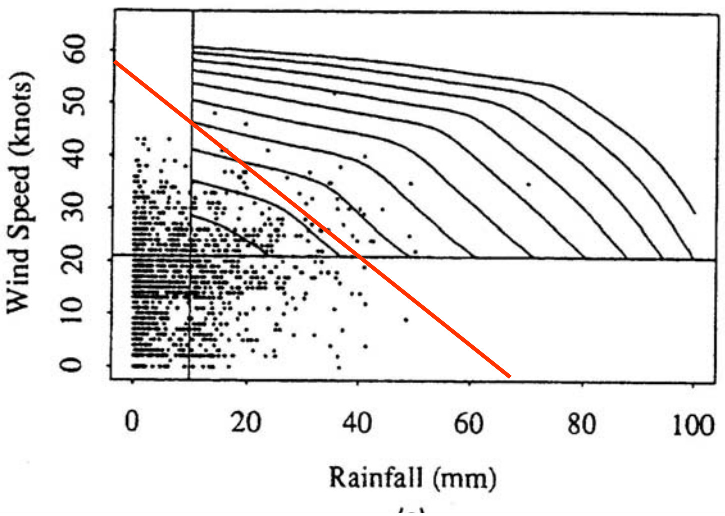
(b) Estimation of Multivariate Extremes

Relevant if structure threatened by a combination of large values of different variables



Ledford & Tawn (1997, 1998, 2003): estimation methods based on tail dependence models.

Example: from study for reservoir flood safety (DoE, Inst of Civ Eng)



Tail dependence model

Heffernan & Tawn (2003): powerful methods based on *conditional distribution* of components of a random vector given that at least one component is large.

(c) Model-Data Fusion for Extremes

Observational evidence

Model evidence

Comparison: model predictions vs observations helps us learn about model uncertainty and predictive reliability.

Systematize and quantify model uncertainty and reliability by treating model inadequacy as a random entity

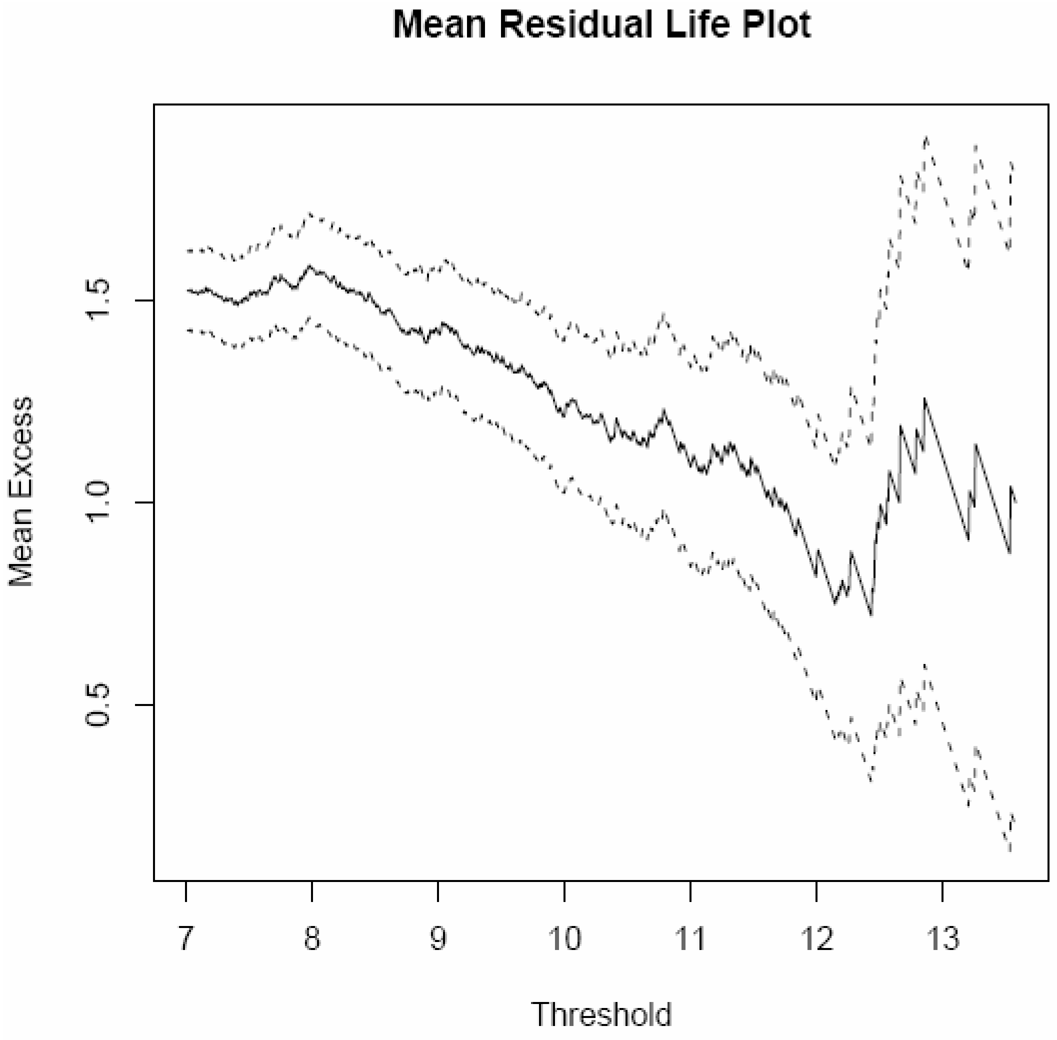
$$\text{data} = \text{model prediction} + \text{error}$$

model inadequacy
Gaussian process?

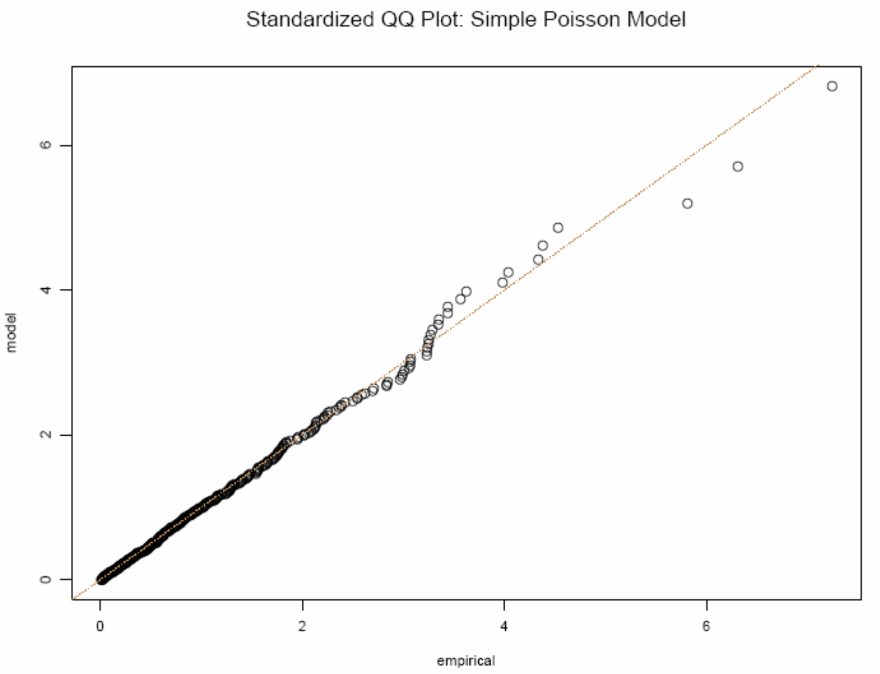
about which we update our knowledge in the light of model-data comparisons. Bayesian methods give a natural approach.

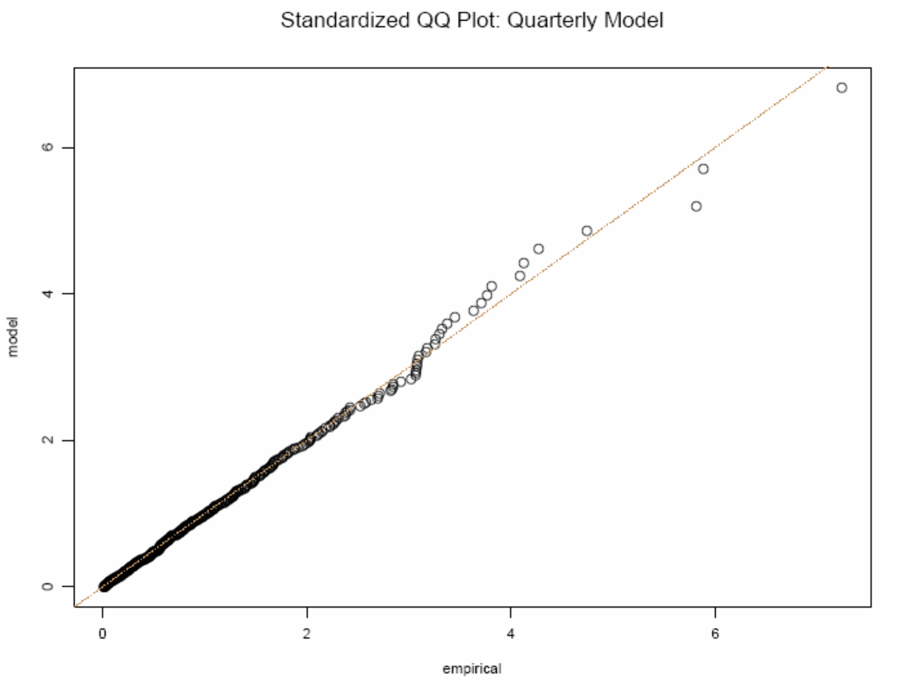
Being developed in programme *Managing Uncertainty in Complex Models (MUCM)* for non-extreme observations.

Atlantic 5421: Hs Mean Excess Plot



Atlantic 5421: QQ Plots for Threshold Models - Stationary and Seasonal





Atlantic 5421: RL CIs for a Seasonal and a Non-Seasonal model